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Interactions between Algebraic Geometry and Noncommutative Algebra

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Abstract. The workshop presented the current developments in the field of noncommutative algebra geometry and its interactions with algebraic geometry and representation theory.

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Introduction by the Organisers

This meeting had 53 participants from 12 countries (Belgium, Canada, Czechoslovakia, China, France, Germany, Japan, Norway, Russia, Switzerland, UK and the US) and 24 lectures were presented during the five day period. The sponsorship of the European Union and other organizations allowed the organizers to invite and secure the participation of a number of young investigators, some of whom presented thirty-minute lectures. As always, there was a substantial amount of mathematical activity outside the formal lecture sessions. This meeting explored the applications of ideas and techniques from algebraic geometry to noncommutative algebra and vice-versa. A number of lectures presented open problems. Areas covered include

- noncommutative projective algebraic geometry,
- (quantized) quiver varieties/quiver representations,
- deformation theory,
- representation theory of Cherednik and related Hecke algebras
- $\mathcal{D}$-module theory,

A number of advances in the above areas were presented and possible starting points for further research proposed. The breadth of the conference is illustrated by the abstracts.

Workshop: Interactions between Algebraic Geometry and Noncommutative Algebra

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Abstracts

A counterexample to the Poisson Dixmier-Moeglin equivalence

Jason Bell
(joint work with Stéphane Launois, Omar Leon-Sanchez, Rahim Moosa)

Let A be a finitely generated complex algebra that is equipped with a Poisson bracket $\{\,,\,,\,\}: A \times A \to A$; that is, $\{\,,\,,\,\}$ is a Lie bracket and for every $a \in A$, we have $\{a, -\}$ is a derivation of A. An ideal I of A is called a *Poisson ideal* if $\{I, A\} \subseteq A$. Then we say that A is Poisson algebra. Brown and Gordon asked whether the Poisson Dixmier-Moeglin equivalence holds for prime Poisson ideals in any complex affine Poisson algebra; that is, whether the sets of Poisson rational ideals, Poisson primitive ideals, and Poisson locally closed ideals coincide. Here, we recall that a prime ideal P is Poisson locally closed if it is locally closed in the the space of Poisson prime ideals (with topology given by the induced topology by the Zariski topology); P is Poisson rational if the induced Poisson bracket on the field of fractions of A/P has the property that if $f \in \mathrm{Frac}(A/P)$ has the property that $\{f, x\} = 0$ for all $x \in A/P$ then f is in $\mathbb{C}$; finally P is Poisson primitive if there is a maximal ideal M that contains P such that P is the largest Poisson ideal contained in M.

We give a complete answer is given to this question using techniques from differential-algebraic geometry and model theory. In particular, it is shown that while the sets of Poisson rational and Poisson primitive ideals coincide, in every Krull dimension at least four there are complex affine Poisson algebras with Poisson rational ideals that are not Poisson locally closed. A weaker version of the Poisson Dixmier-Moeglin equivalence is proven for all complex affine Poisson algebras, from which it follows that the full equivalence holds in Krull dimension three or less. Finally, we show that everything, except possibly that rationality implies primitivity, can be done over an arbitrary base field of characteristic zero.

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Caloger-Moser Spaces and Grothendieck's Ring of Varieties

Gwyn Bellamy

Associated to a smooth curve C, finite group G of automorphisms of C, and a positive integer n is the *Calogero-Moser deformation* $\mathsf{CM}_n(C, G) \to \mathfrak{c}_n$. It is a flat Poisson deformation of $T^*(C^n/\mathfrak{S}_n \wr G)$ over the affine base $\mathfrak{c}_n$, where $\mathfrak{S}_n$ is the symmetric group. The deformation can be constructed as a certain coarse moduli space. We would like to understand these spaces better. A first step in doing so is to consider, for each $\mathbf{c} \in \mathfrak{c}_n$, the class of the fiber $\mathsf{CM}_{n,\mathbf{c}}(C,G)$ in Grothendieck's ring of varieties $K_0(\mathrm{Var}_{\mathbb{C}})$. We show that, for $\mathbf{c}$ Weil generic,

$$\sum_{n\geq 0}[\mathsf{CM}_{n,\mathbf{c}}(C,G)]t^n = \left(\prod_{n\geq 1}\frac{1}{1-\mathbb{L}^n t^n}\right)^{[C/G]}$$

where $\mathbb{L}$ is the Lefschetz motive. This implies that $[\mathsf{CM}_{n,\mathbf{c}}(C,G)]$ equals the class of the Hilbert scheme $\mathrm{Hilb}^n(T^*(C/G))$ of n points in $T^*(C/G)$ in $K_0(\mathrm{Var}_{\mathbb{C}})$. For arbitrary parameters $\mathbf{c}$, we present a conjectural formula for the power series $\sum_{n\geq 0}[\mathsf{CM}_{n,\mathbf{c}}(C,G)]t^n$ in terms of certain generalized characters of higher level Fock spaces for the quantum group $U_q(\widehat{\mathfrak{sl}}_\infty)$. This talk is based on joint work in progress with Oleg Chalykh.

Geometric Tilting Objects

Ragnar-Olaf Buchweitz

After reviewing the definitions and structure theorem for d-representation infinite and $(d+1)$-higher preprojective algebras (due to Keller, Herschend-Iyama-Oppermann, Amiot-Iyama-Reiten, Minamoto-Mori) we discussed the notion of geometric objects in *Ext*-finite triangulated categories with Serre functor starting from a suggestion by Bondal.
We say that an object F in such a category has Serre dimension d if there exists an $a \geq 0$ such that for each $i \in \mathbb{Z}$,

$$Ext^j(S_d^i F, F) = 0 \text{ for } j \notin [-a, a+d]$$

where $S_d := S \circ [-d]$ is the translated Serre functor. The object is sheaf-like if one can take $a = 0$. It T is a tilting object with endomorphism algebra Λ , then T has Serre dimension $d = gldim(\Lambda)$ and is sheaf-like if and only if Λ is d-representation infinite and then the associated $(d+1)$-preprojective algebra is $\oplus_{i\geq 0}Hom(S_d^i T, T)$ which one can interpret as the pullback of T to the (virtual) canonical bundle still being a tilting object. If the $(d+1)$-preprojective algebra is graded coherent (as is conjectured to always hold), then the work of Miramoto implies that $D^b(\Lambda)$ carries a geometric t-structure of dimension d and thus, satisfies Bondal's requirement

Construction of $\mathcal{D}$-Modules with prescribed p-support

Christopher Dodd

In this talk, I discussed the construction of (a part of) the *quantization correspondence* for algebraic $\mathcal{D}$-modues motivated by conjectures of Kontsevich and Belov-Kanel. We explained how, to any smooth Lagrangian $\mathcal{L} \subset T^*X$ (where X is a smooth complex affine variety) with $H^1(\mathcal{L}, \mathbb{C}) = 0$, one can associate in a natural way a holonomic $\mathcal{D}_X$-module $\mathcal{M}$. This $\mathcal{D}_X$-module is irreducible and it has the propert that after reduction mod p its "p-support" is equal to $\mathcal{L}_k^{(1)} \subset T^*X_k^{(1)}$ where k is a perfect field of positive characteristic for al $p \gg 0$. If $H^1(\mathcal{L}, \mathbb{Z}) = 0$, this module is unique. As a consequence, we explained that this implies the existence of an isomorphism

$$Aut_{Mor}(\mathcal{D}_n) \xrightarrow{\sim} Aut_{Symp}(T^*\mathbb{A}^n)$$

which is a weak version of a very surprising conjectuve if Kotsevich and Belov-Kanel

Cohomology of Non-Commutative Hilbert Schemes as Modules over the Cohomological Hall Algebra

Hans Franzen

1. Non-Commutative Hilbert Schemes

Non-commutative Hilbert schemes generalize classical Hilbert schemes in the following way: The Hilbert scheme $\mathrm{Hilb}_d(\mathbb{A}^m)$ is a variety whose (complex valued) points parametrize ideals I of the polynomial algebra $\mathbb{C}[x_1, \dots, x_m]$ such that $\mathbb{C}[x_1, \dots, x_m]/I$ has dimension d as a complex vector space.

In contrast, the *non-commutative* Hilbert scheme $\mathrm{Hilb}_d^{(m)}$ is a variety whose points are in bijection to *left* ideals I of the free non-commutative algebra $A := \mathbb{C}\langle x_1, \dots, x_m\rangle$ in m variables having codimension d (as a complex vector space).

A result of Nakajima [4] asserts that the direct sum $\bigoplus_d H^*(\mathrm{Hilb}_d(\mathbb{A}^2))$ of the (singular) cohomology groups (with rational coefficients) carries the structure of a representation of the infinite-dimensional Heisenberg algebra; in fact, it is isomorphic to the bosonic Fock space. Using this structure, Lehn and Sorger (cf. [3]) were able to identify the cohomology *rings* $H^*(\mathrm{Hilb}_d(\mathbb{A}^2))$ with the ring of class functions of the symmetric group S_d. We would like to find a suitable analog of these results for the cohomology of the non-commutative Hilbert schemes $\mathrm{Hilb}_d^{(m)}$.

The variety $\mathrm{Hilb}_d^{(m)}$ can be constructed using geometric invariant theory. Let

$$\hat{R}_d := \mathbb{C}^d \times R_d$$

where $R_d := M_{d\times d}(\mathbb{C})^m$. The vector space $\hat{R}_d$ is equipped with an action of the general linear group Gl_d by left multiplication on the vector and simultaneous conjugation on the matrices. We take the stability condition induced by the determinantal character of Gl_d. This means that a point (v, ϕ_*) is (semi-)stable if and only if v generates the d-dimensional left-A-module which arises by the m-tuple of matrices ϕ_*. The geometric quotient

$$\hat{R}_d^{\mathrm{st}} / \mathrm{Gl}_d,$$

which is even a principal Gl_d-fiber bundle, is isomorphic to $\mathrm{Hilb}_d^{(m)}$ by identifying the orbit of a stable point (v, ϕ) with the left-ideal $\mathrm{Ann}_A(v)$. Note that $\mathrm{Hilb}_d^{(m)}$ is a smooth variety.

2. Cohomological Hall Algebra

The Cohomological Hall algebra, which we will call CoHa in the following, was invented by Kontsevich and Soibelman (cf. [2]). It was constructed in order to provide a categorification of (motivic) Donaldson-Thomas invariants. Their construction works in vast generality; we are considering the CoHa $\mathcal{H}$ of the m-loop quiver (and trivial potential). As a graded vector space, it is defined as direct sum

$$\mathcal{H} = \bigoplus_{d\geq 0} H^*_{\mathrm{Gl}_d}(R_d)$$

of equivariant singular cohomology groups with rational coefficients. The multiplication is induced by suitable "Hecke correspondences"

$$R_p \times R_q \leftarrow R_{p,q} \rightarrow R_{p+q},$$

where $R_{p,q}$ is the vector space of m-tuples of $(d\times d)$-matrices having a block upper triangular shape $\left(\begin{smallmatrix} p & * \\ & q \end{smallmatrix}\right)$. Kontsevich and Soibelman give an algebraic description of the thus obtained multiplication by identifying

$$H^*_{\mathrm{Gl}_d}(R_d) \cong H^*_{\mathrm{Gl}_d}(\mathrm{pt}) \cong H^*_{T_d}(\mathrm{pt})^{S_d} \cong \mathrm{Sym}(X(T_d))^{S_d} \cong \mathbb{Q}[x_1,\ldots,x_d]^{S_d}.$$

Using Bott localization, they prove:

Theorem 1 (Kontsevich-Soibelman)**.** *For $f\in\mathcal{H}_p$ and $g\in\mathcal{H}_q$, the product $f*g$, regarded as a symmetric polynomial in $d=p+q$ variables, equals*

$$\sum_{\substack{1\leq i_1<\ldots<i_p\leq d\\ 1\leq j_1<\ldots<j_q\leq d\\ \textit{complementary}}} f(x_{i_1},\ldots,x_{i_p})g(x_{j_1},\ldots,x_{j_q})\prod_{\mu=1}^{p}\prod_{\nu=1}^{q}(x_{j_\nu}-x_{i_\mu})^{m-1}.$$

3. A CoHa-Module

We are going to define a left-$\mathcal{H}$-module structure on the direct sum

$$\bigoplus_{d\geq 0} H^*(\mathrm{Hilb}_d^{(m)})$$

using the fact that $H^*(\mathrm{Hilb}_d^{(m)}) = H^*_{\mathrm{Gl}_d}(\hat{R}_d^{\mathrm{st}})$. We work with a modified version of the Hecke correspondences

$$R_p \times \hat{R}_q^{\mathrm{st}} \leftarrow \hat{R}_{p,q}^{\mathrm{st}} \rightarrow \hat{R}_{p+q}^{\mathrm{st}},$$

where $\hat{R}_{p,q}^{\mathrm{st}}$ is by definition $(\mathbb{C}^d\times R_{p,q})\cap\hat{R}_d^{\mathrm{st}}$. These types of CoHa-modules arising from stable framed objects are also considered in [6] in greater generality. The equivariant maps $R_d \leftarrow \hat{R}_d \hookleftarrow \hat{R}_d^{\mathrm{st}}$ induce a map

$$j^* : \mathcal{H} \rightarrow \bigoplus_{d\geq 0} H^*(\mathrm{Hilb}_d^{(m)})$$

which can easily seen to be $\mathcal{H}$-linear (with $\mathcal{H}$ considered as a left-module over itself) and surjective. In [1], we show:

Theorem 2. *The kernel of j^* equals*

$$\bigoplus_{p\geq 0, q>0} \mathcal{H}_p * (e_q \cup \mathcal{H}_q),$$

where $*$ is the CoHa-multiplication, $e_q(x_1, \ldots, x_q) = x_1 \ldots x_q$, and $\cup$ is the cup product on $\mathcal{H}_q = H^*_{\mathrm{Gl}_q}(R_q)$ which identifies with the multiplication of symmetric polynomials.

The proof of the above theorem relies on Harder-Narasimhan methods and computations with fiber bundles. It can be generalized to modules over the CoHa of an arbitrary quiver arising from arbitrary framing data. As a consequence of Theorem 2, we obtain a description of the cohomology *ring* $H^*(\mathrm{Hilb}_d^{(m)})$ in terms of generators and relations which can be regarded as an analog of Lehn-Sorger's result. Choose a basis of $\mathcal{H}_p \otimes \mathcal{H}_q$ over $\mathcal{H}_d$ whose elements we may assume to be of the form $f_{\lambda,p} \otimes g_{\lambda,q}$.

Corollary 3. *The kernel of the map $j^* : \mathbb{Q}[e_1, \ldots, e_d] \to H^*(\mathrm{Hilb}_d^{(m)})$ is generated by expressions of the form*

$$f_{\lambda,p} * (e_q \cup g_{\lambda,q})$$

with $p = 0, \ldots, p-1$, $q = d-p$ and $\lambda = 1, \ldots, \binom{d}{p}$.

The non-commutative Hilbert scheme $\mathrm{Hilb}_d^{(m)}$ is a fine moduli space which means it possesses a universal bundle $\mathcal{U}$ of rank d. Under the map j^*, the generator e_i is sent to $c_i(\mathcal{U})$. Using a cell decomposition of Reineke (cf. [5]), we obtain a monomial basis in Chern classes of $\mathcal{U}$ for the cohomology with *integral* coefficients.

Proposition 4. *The cohomology $H^*(\mathrm{Hilb}_d^{(m)}; \mathbb{Z})$ with integral coefficients is a free abelian group with a basis given by the monomials*

$$c_1(\mathcal{U})^{b_{d-1}} \ldots c_{d-1}(\mathcal{U})^{b_1}$$

where $b = (b_1, \ldots, b_d)$ is a tuple of non-negative integers satisfying $b_1 + \ldots + b_i \leq (m-1)d$ for every $i = 1, \ldots, d-1$.

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Quantum cluster algebras and quantum nilpotent algebras

K. R. Goodearl
(joint work with M.T. Yakimov)

The theme of this talk was general quantum cluster algebra structures and their construction within the large, axiomatically defined class of quantum nilpotent algebras. The approach was developed in [1, 2]. Many important families of quantized coordinate rings are subsumed in the class covered by these methods, such as that of quantum Schubert cell algebras. In particular, the results extend the theorem of Geiss, Leclerc and Schröer for the case of symmetric Kac–Moody groups. A consequence is the verification of the Berenstein–Zelevinsky conjecture postulating the existence of quantum cluster algebra structures on all quantized coordinate rings of double Bruhat cells in simple Lie groups.

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Highest weight and monoidal structure for strict polynomial functors

Henning Krause

In representation theory of artin algebras, it is an old idea of Auslander [3] to 'resolve' an algebra A by finding an algebra B of finite global dimension together with an idempotent e such that eBe is isomorphic to A. Later, this idea was refined by asking for the algebra B to be quasi-hereditary. A prototypical example is the pair (A, B) consisting of the group algebra A of the symmetric group and B the corresponding Schur algebra. In fact, the right perspective seems to be the study of the category of strict polynomial polynomial functors (in the sense of Friedlander–Suslin [5]), which is equivalent to modules over a Schur algebra and contains the representation theory of the symmetric group.

In some recent work [7] the highest weight structure for the category of strict polynomial functors is explained in elementary terms, using classical facts about Schur functors [1] (including the Cauchy decomposition formula) and working over an arbitrary commutative ring. Of course, this implies the well-known fact that Schur algebras are quasi-hereditary and have finite global dimension. The point here is to offer an elementary approach which should help to understanding more interesting situations, as they arise for instance in recent work by Buchweitz–Leuschke–Van den Bergh [4].

The second aspect of this work concerns the monoidal structure. There is a tensor product for the category of strict polynomial functors which one can use to describe the Koszul-Ringel duality [6]. Recent work of Aquilino–Reischuk [2] shows that this tensor product is actually compatible with the tensor product for representations of the symmetric group. Thus we have a resolution (in Auslander's sense) which is in fact a monoidal functor.

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Irreducible components of varieties of representations

Birge Huisgen-Zimmermann

Let A be a finite dimensional algebra over an algebraically closed field. In this general setting, we "corner" the irreducible components of the varieties $\mathrm{Rep}_{\mathbf{d}}A$ parametrizing the A-modules with fixed dimension vector $\mathbf{d}$ through upper semicontinuous maps from $\mathrm{Rep}_{\mathbf{d}}A$ to suitable posets. This process leads to a finite collection of closed irreducible subvarieties of $\mathrm{Rep}_{\mathbf{d}}A$, which includes the irreducible components. The subsequent process of sifting is, however, not expected to allow for a satisfying general solution, but needs to be customized for special classes of algebras. The first nontrivial instances of a full listing of the components were provided by Schröer (with different methods) for the class of Gelfand-Ponomarev algebras, a good illustration of the intricacy of the task.

We show how, for truncated path algebras of arbitrary quivers, the above strategy allows to detect and separate the components in terms of generic invariants of their modules. We are thus led to easily applicable criteria for determining the components from quiver and Loewy length of A; the components are paired with an array of representation-theoretic properties of their modules. In the special case where A is local (that is, based on a quiver with a single vertex and a finite number r of loops), the component problem is particularly easily resolved. In particular, it is easy to count the number of components for any dimension d as a function of r and the Loewy length of A. The latter counts generalize existing formulas for $r = 2$ and Loewy length 2, first obtained by Donald and Flanigan.

Cyclotomic Mackey Functors

Dmitry Kaledin

In noncommutative geometry, differential forms and de Rham cohomology classes appear in the form of Hochschild and cyclic homology. The corresponding natural linear algebraic structures such as the de Rham differential and the Hodge filtration are conveniently packaged using the category Λ introduced by A. Connes. However, for algebraic varieties over a finite field or a ring of p-adic integers, de Rham cohomology can be refined to the cristalline cohomology of A. Grothendieck and all parts of the story have their counterparts such as the de Rham-Witt complex of P. Deligne and L. Illusie and the filtered Dieudonné module structure on cohomology in the sense of J-M. Fontaine. If this refinement were to extend to the noncomutative setting, one would need a way to package additional linear algebra in a sufficiently simple and natural way, possibly extending Connes' approach. The notion of a cyclotomic Mackey functor is designed to give such a packaging. Its definition combines Connes' cyclic objects and the so-called Mackey functors for the group $\mathbb{Z}$ and automatically provides al the linear algebra one needs.

Quantized Gieseker moduli spaces

Ivan Losev

This talk describes my recent results on the representation theory of quantized Gieseker moduli spaces, [2]. Let us define those quantized spaces.

We start with a pair n, r of positive integers. Set $V := \mathbb{C}^n, W := \mathbb{C}^r$. We form the space $R := \mathfrak{sl}(V) \oplus \operatorname{Hom}(V, W)$, this space comes equipped with a natural action of the group $G = \operatorname{GL}(V)$. We have an induced action of G on $T^*R = R \oplus R^* \cong \mathfrak{sl}(V)^{\oplus 2} \oplus \operatorname{Hom}(V, W) \oplus \operatorname{Hom}(W, V)$. This action is Hamiltonian with moment map $\mu : T^*R \to \mathfrak{g}^* \cong \mathfrak{g}$ given by $\mu(A, B, i, j) := [A, B] + ji$. The dual map $\mu^* : \mathfrak{g} \to \mathbb{C}[T^*R]$ is given by $x \mapsto x_R$, where x_R denotes the vector field on R induced by x, this vector field is viewed as a function on T^*R.

Using the Hamiltonian action, we can form several different versions of a Hamiltonian reduction. First, set $\mathcal{M}(n, r) := \mu^{-1}(0)//G$, i.e., $\mathcal{M}(n, r)$ is an affine algebraic variety with $\mathbb{C}[\mathcal{M}(n, r)] = (\mathbb{C}[T^*R]/\mathbb{C}[T^*R]\{x_R | x \in \mathfrak{g}\})^G$. Next, pick a nonzero integer θ that can be viewed as a character of G via the identification $\mathbb{Z} \xrightarrow{\sim} \operatorname{Hom}(G, \mathbb{C}^\times), \theta \mapsto [g \mapsto \det(g)^\theta]$. So we can form the GIT quotient $\mathcal{M}^\theta(n, r) := \mu^{-1}(0)^{\theta-ss}//G$. The product of $\mathcal{M}^\theta(n, r)$ with $\mathbb{C}^2$ is the Gieseker moduli space parameterizing the torsion free rank r degree n sheaves on $\mathbb{P}^2$ trivialized at infinity. Finally, we can form the quantum Hamiltonian reduction quantizing $\mathcal{M}(n, r)$. Namely, pick $\lambda \in \mathbb{C}$ and view λ as the character $x \mapsto \lambda \operatorname{tr}(x)$ of $\mathfrak{g}$. The quantum Hamiltonian reduction $\mathcal{A}_\lambda(n, r)$ is, by definition, $(D(R)/D(R)\{x_R - \langle \lambda, x \rangle, x \in \mathfrak{g}\})^G$, where $D(R)$ stands for the algebra of linear differential operators on R. The algebra $\mathcal{A}_\lambda(n, r)$ is filtered and the associated graded coincides with $\mathbb{C}[\mathcal{M}(n, r)]$. We could also form a quantization $\mathcal{A}^\theta_\lambda(n, r)$ of

$\mathcal{M}^\theta(n,r)$ that is a sheaf of filtered algebras playing a crucial role in the theory but we are not going to do that in order to keep the exposition simple.

The goal of this project is to study the representation theory of $\mathcal{A}_\lambda(n,r)$. One motivation is an application to the representation theory of the quantizations of more general Nakajima quiver varieties. For example, in [2], results on $\mathcal{A}_\lambda(n,r)$ were applied to extend the main theorem of [1] on the number of finite dimensional irreducible representations of quantized quiver varieties to the case of affine type quivers with arbitrary framing.

Let us list some results on the representation theory of $\mathcal{A}_\lambda(n,r)$.

Theorem 1. *The algebra $\mathcal{A}_\lambda(n,r)$ has a finite dimensional irreducible representation if and only if λ is a rational number of the form $\frac{q}{n}$ with* $\mathrm{GCD}(q,n)=1$ *lying outside the interval $(-r,0)$. If λ has this form, then there is a unique finite dimensional irreducible representation and every finite dimensional representation is completely reducible.*

Theorem 2. *The algebra $\mathcal{A}_\lambda(n,r)$ has infinite homological dimension if and only if $\lambda \in (-r,0)$ is a rational number with denominator $\leq n$.*

Theorem 3. *Suppose $\mathcal{A}_\lambda(n,r)$ has finite homological dimension and let n' denote the denominator of λ. Then there are precisely $[n/n']$ proper two-sided ideals in $\mathcal{A}_\lambda(n,r)$. They form a chain and all of them are prime.*

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Hochschild cohomology with support

Wendy Lowen

In this talk, we present our recent work [8], in which we propose a new point of view in order to treat and obtain certain decompositions for Hochschild cohomology. We illustrate this by revisiting several results on Hochschild cohomology of schemes which were obtained in joint work with Michel Van den Bergh [10],[11].

0.1. **Hochschild cohomology.** Hochschild cohomology is originally a cohomology theory for algebras. For a k-algebra A (with k a field), we have

$$HH^n(A) = Ext^n_{A-A}(A,A)$$

computed in the category of A-bimodules. It is the cohomology of the Hochschild complex $\mathbf{C}(A)$, which is a B_∞-algebra, in particular it is endowed with all the algebraic structure relavant for higher order deformation theory. In general, the cohomology is notoriously hard to compute, although in some special cases, computations can be carried out:

(1.a) For a commutative affine regular algebra A, the classical Hochschild-Kostant-Rosenberg theorem states that $HH^n(A) = \Lambda^n_A \mathrm{Der}(A)$.

(1.b) For an upper triangular matrix algebra $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, we have an associated Happel long exact sequence [4]

$$\dots \longrightarrow HH^i(R) \longrightarrow HH^i(A) \oplus HH^i(B) \longrightarrow Ext^i_{A-B}(M,M) \longrightarrow \dots$$

Hochschild cohomology has also been considered for a (quasi-compact, separated) scheme X by Kontsevich, Swan, Yekutieli and others, using the following expression:

$$HH^n(X) = Ext^n_{X\times X}(\mathcal{O}_X, \mathcal{O}_X).$$

From this definition, it is not clear whether there exists a B_∞-algebra $\mathbf{C}(X)$ which computes the cohomology. One possible approach to obtain such a B_∞-algebra follows from the work of Gerstenhaber and Schack in two steps:

(2.a) These authors define Hochschild cohomology for an arbitrary presheaf $\mathcal{A}$ of algebras as $HH^n(\mathcal{A}) = Ext^n_{\mathcal{A}-\mathcal{A}}(\mathcal{A}, \mathcal{A})$, and for a scheme X, they propose $HH^n(\mathcal{A})$, where $\mathcal{A}$ is the structure sheaf restricted to an affine basis, as the correct cohomology of X [2]. One can show that $HH^n(X) = HH^n(\mathcal{A})$ [10].

(2.b) For a presheaf of algebras $\mathcal{A}$, these authors construct a single algebra A with $HH^n(\mathcal{A}) \cong HH^n(A)$ [3]. Thus, we arrive at the Hochschild complex $\mathbf{C}(A)$ as a B_∞-algebra computing $HH^n(X)$.

0.2. **Map-graded categories.** The aim of this talk is to present a general framework in which to understand results like (1.b) and (2.b), and in which to state some general decomposition techniques for Hochschild complexes. These techniques are of a different nature than the classical HKR type decompositions, and do not rely on assumptions like smoothness or characteristic zero.

The fundamental objects in our approach are *map-graded categories* [6]. A map-graded category $\mathfrak{a}$ can be viewed as a monoid-graded algebra with several objects. It has an underlying grading category $\mathcal{U}$, object sets $\mathfrak{a}_U$ for $U \in \mathcal{U}$, and morphism modules $\mathfrak{a}_u(A, A')$ for $u : U \longrightarrow U'$ in $\mathcal{U}$, $A \in \mathfrak{a}_U$ and $A' \in \mathfrak{a}_{U'}$. These data should satisfy the obvious category-type axioms. The following are examples of map-graded categories:

(3.a) If $\mathcal{U}$ satisfies $\mathcal{U}(U, U') = \{*\}$ or $\mathcal{U}(U, U') = \emptyset$ for all $U, U' \in \mathcal{U}$, then $\mathcal{U}$-graded categories are in 1-1 correspondence with linear categories $\mathfrak{a}$ with $\mathrm{Ob}(\mathfrak{a}) = \mathcal{U}$ and $\mathfrak{a}(U, U') = 0$ if $\mathcal{U}(U, U') = \emptyset$. Thus, in this case a grading prescribes a certain "shape" for a linear category. This situation was described in [10] using so called censoring relations on a category.

(3.b) For a presheaf of k-algebras $\mathcal{A} : \mathcal{U} \longrightarrow \mathsf{Alg}(k)$, there is an associated $\mathcal{U}$-graded category $\mathfrak{a}$ obtained as a kind of k-linear Grothendieck construction from $\mathcal{A}$ in the spirit of [1] (where the input is a pseudofunctor $\mathcal{U} \longrightarrow \mathsf{Cat}$ and the output is a category fibered over $\mathcal{U}$). The category $\mathfrak{a}$ is an intermediate object between $\mathcal{A}$ and the algebra A from (2.b).

(3.c) For two linear categories $\mathfrak{a}$ and $\mathfrak{b}$ and an $\mathfrak{a}$-$\mathfrak{b}$-bimodule M, Keller's *arrow category* $(\mathfrak{b} \rightarrow_M \mathfrak{a})$ [5] is naturally graded over the path category of $\bullet \rightarrow \bullet$. It is the natural categorical version of the algebra R from (1.b).

Small map-graded categories (over a fixed k) can be organized into a category Map. For a functor $\varphi : \mathcal{V} \longrightarrow \mathcal{U}$ between small categories and a $\mathcal{U}$-graded category $\mathfrak{a}$, we obtain a naturally induced $\mathcal{V}$-graded category $\mathfrak{a}^{\varphi}$ with $\mathfrak{a}^{\varphi}_V = \mathfrak{a}_{\varphi(V)}$ for $V \in \mathcal{V}$ and similarly for morphisms. This way, Map becomes fibered over the category Cat of small categories through the functor

$$\mathsf{Map} \longrightarrow \mathsf{Cat} : (\mathcal{U}, \mathfrak{a}) \longmapsto \mathcal{U}.$$

0.3. **Limited functoriality.** An important shortcoming of Hochschild cohomology of algebras is it's lack of functoriality, which is known to be remedied in part by turning to linear categories [5]. Our point of view is that limited functoriality is in fact fundamentally determined by grading categories.

Let $\mathfrak{a}$ be a $\mathcal{U}$-graded category. Let the nerve $\mathcal{N}(\mathfrak{a})$ of $\mathfrak{a}$ be the simplicial set with n-simplices $\sigma = (u, A)$ given by data

$$\begin{array}{ccccccc} A_0 & & A_1 & & \dots & & A_n \\ & & & & & & \\ U_0 & \xrightarrow[u_0]{} & U_1 & \xrightarrow[u_1]{} & \dots & \xrightarrow[u_{n-1}]{} & U_n \end{array}$$

with $u_i \in \mathcal{U}, A_i \in \mathfrak{a}_{U_i}$.

The *Hochschild complex of* $\mathfrak{a}$ is the complex $\mathbf{C}_{\mathcal{U}}(\mathfrak{a})$ with

$$\mathbf{C}^n_{\mathcal{U}}(\mathfrak{a}) = \prod_{(u,A) \in \mathcal{N}(\mathfrak{a})} Hom_k(\mathfrak{a}_{u_{n-1}}(A_{n-1}, A_n) \otimes \dots \otimes \mathfrak{a}_{u_0}(A_0, A_1), \mathfrak{a}_{u_{n-1} \cdots u_1 u_0}(A_0, A_n))$$

with the simplicial Hochschild differential. This complex is in fact a B_∞-algebra [6].

Let $\mathsf{Map}_c \subseteq \mathsf{Map}$ denote the full subcategory of cartesian morphisms with respect to the fibered category $\mathsf{Map} \longrightarrow \mathsf{Cat}$. It is easily observed that taking Hochschild complexes defines a functor

$$\mathbf{C} : \mathsf{Map}_c \longrightarrow B_\infty : (\mathcal{U}, \mathfrak{a}) \longmapsto \mathbf{C}_{\mathcal{U}}(\mathfrak{a}).$$

Example 1. *Consider an arrow category* $(\mathfrak{b} \rightarrow_M \mathfrak{a})$ *as in (3.c). The functor* $(\bullet \quad \bullet) \longrightarrow (\bullet \rightarrow \bullet)$ *underlies a cartesian morphism* $\mathfrak{b} \coprod \mathfrak{a} \longrightarrow \mathfrak{b} \rightarrow_M \mathfrak{a}$. *Functoriality yields an exact sequence*

$$0 \longrightarrow \mathbf{C}(\mathfrak{b}; \mathfrak{a}; M) \longrightarrow \mathbf{C}(\mathfrak{b} \rightarrow_M \mathfrak{a}) \longrightarrow \mathbf{C}(\mathfrak{b}) \oplus \mathbf{C}(\mathfrak{a}) \longrightarrow 0$$

in which the kernel $\mathbf{C}(\mathfrak{b}; \mathfrak{a}; M)$ *is seen to compute* $Ext_{\mathfrak{b}-\mathfrak{a}}(M, M[-1])$. *The resulting long exact cohomology sequence generalizes Happel's sequence from (1.b).*

Example 2. *Let* X *be a quasi-compact separated scheme with an affine open basis* $\mathcal{B}$. *By (2.a), we have* $HH^n(X) = Ext^n_{\mathcal{O}_\mathcal{B}-\mathcal{O}_\mathcal{B}}(\mathcal{O}_\mathcal{B}, \mathcal{O}_\mathcal{B})$. *By the Cohomology Comparison Theorem from* [11], *this cohomology is computed by the complex* $\mathbf{C}_\mathcal{B}(\mathfrak{o}_\mathcal{B})$ *where* $\mathfrak{o}_\mathcal{B}$ *is the Grothendieck construction of* $\mathcal{O}_\mathcal{B}$ *as in (3.b). We thus interpret this complex as the Hochschild complex* $\mathbf{C}_\mathcal{B}(X)$ *of* X *with respect to the basis* $\mathcal{B}$. *For an inclusion* $V \subseteq U$ *between open subsets* $U, V \subseteq X$, *let* $\mathcal{B}_U$ *and* $\mathcal{B}_V$ *be the* *restrictions of $\mathcal{B}$ to subsets within U and V respectively. By functoriality, the inclusion $\mathcal{B}_V \subseteq \mathcal{B}_U$ yields $\mathbf{C}_{\mathcal{B}_U}(U) \longrightarrow \mathbf{C}_{\mathcal{B}_V}(V)$. This approach remedies the lack of natural maps $\mathbf{C}(\mathcal{O}(U)) \longrightarrow \mathbf{C}(\mathcal{O}(V))$.*

0.4. **Sheaves and Mayer Vietoris sequences.** In order to make maximal use of grading categories, we now introduce a family of Grothendieck pretopologies on Cat.

Definition 1. *Let $n \in \mathbb{N} \cup \{\infty\}$. A collection of functors $(\varphi_i : \mathcal{V}_i \longrightarrow \mathcal{U})_{i \in I}$ is an n-*cover* provided that it induces jointly surjective collections of maps $(\mathcal{N}(\varphi_i) : \mathcal{N}_k(\mathcal{V}_i) \longrightarrow \mathcal{N}_k(\mathcal{U}))_{i \in I}$ between k-simplices of the simplicial nerves for all $k \leq n$.*

For each $n \in \mathbb{N} \cup \{\infty\}$, we thus obtain a pretopology of n-covers on Cat, and an induced pretopology of n-covers on Map_c. We have:

Theorem 4.
(1) *The category Map constitutes a stack on Cat for each pretopology of n-covers with $n \geq 3$.*
(2) *The functor $\mathbf{C} : \mathsf{Map}_c \longrightarrow B_\infty$ is a sheaf for the pretopology of ∞-covers on Map.*

Example 3. *Consider a $\mathcal{U}$-graded category $\mathfrak{a}$ and two subcategories $\varphi_i : \mathcal{V}_i \subseteq \mathcal{U}$ for $i \in \{1, 2\}$ that constitute an ∞-cover of $\mathcal{U}$. Let $(\mathcal{V}_i, \mathfrak{a}^{\varphi_i}) \longrightarrow (\mathcal{U}, \mathfrak{a})$ and $(\mathcal{V}_1 \cap \mathcal{V}_2, \mathfrak{a}^{\varphi}) \longrightarrow (\mathcal{U}, \mathfrak{a})$ be corresponding cartesian morphisms. By the sheaf property of $\mathbf{C}$, we obtain a Mayer-Vietoris sequence of Hochschild complexes:*

$$0 \longrightarrow \mathbf{C}_{\mathcal{U}}(\mathfrak{a}) \longrightarrow \mathbf{C}_{\mathcal{V}_1}(\mathfrak{a}^{\varphi_1}) \oplus \mathbf{C}_{\mathcal{V}_2}(\mathfrak{a}^{\varphi_2}) \longrightarrow \mathbf{C}_{\mathcal{V}_1 \cap \mathcal{V}_2}(\mathfrak{a}^{\varphi}) \longrightarrow 0.$$

Example 4. *Let X and $\mathcal{B}$ be as in Example 2. Let $X = U_1 \cup U_2$ be an open cover and put $U_{12} = U_1 \cap U_2$. Let $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_{12}$ be the restrictions of $\mathcal{B}$ to U_1, U_2 and U_{12} respectively. Put $\mathcal{B}_0 = \mathcal{B}_1 \cup \mathcal{B}_2 \subset \mathcal{B}$. As categories, $\mathcal{B}_1$ and $\mathcal{B}_2$ together constitute an ∞-cover of $\mathcal{B}_0$, whence we obtain the Mayer-Vietoris sequence:*

$$0 \longrightarrow \mathbf{C}_{\mathcal{B}_0}(X) \longrightarrow \mathbf{C}_{\mathcal{B}_1}(U_1) \oplus \mathbf{C}_{\mathcal{B}_2}(U_2) \longrightarrow \mathbf{C}_{\mathcal{B}_{12}}(U_{12}) \longrightarrow 0.$$

A more refined use of bases allows to obtain a sheaf of Hochschild complexes on quasi-compact opens, as was shown in [7]*. On the other hand, the existence of Mayer-Vietoris long exact sequences for arbitrary ringed spaces was shown more generally in* [10]*.*

Example 5. *Let $\mathcal{U}$ be a finite poset and $\mathfrak{a}$ a $\mathcal{U}$-graded category. Looking at a Hasse diagram of $\mathcal{U}$, we can ∞-cover $\mathcal{U}$ "horizontally" by its "vertical" maximal simplices. This covering is such that both the involved simplices and their intersections are path categories of the following type:*

$$\mathcal{V} = \langle \bullet \to \bullet \to \cdots \to \cdots \to \bullet \to \bullet \rangle$$

The Hochschild complex of each induced $\mathcal{V}$-graded category can be decomposed by treating it as an "iterated arrow category". This illustrates the fact that the exact sequences obtained in Example 1 and Example 3 respectively should be seen as complementary tools for deconstructing Hochschild complexes.

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Motivic Donaldson–Thomas invariants of quantum $\mathbb{C}^3$.

Andrew Morrison

(joint work with Brent Pym and Balázs Szendrői)

To the memory of Kentaro Nagao.

Donaldson–Thomas invariants of a projective Calabi–Yau threefold (X, H) were defined in [11] as (virtual) counts of the number of stable sheaves on X. For rank one sheaves with vanishing second Chern character they give one of the ways of counting curves in X.

Motivic DT invariants [4] refine these deformation invariant numbers by associating to each moduli space of stable sheaves it's *virtual motive*: an element of $K_0^{\hat{\mu}}(Var_{\mathbb{C}})[\mathbb{L}^{-\frac{1}{2}}]$ whose Euler characteristic is the classical DT invariant[1]. The refined virtual motive is no longer deformation invariant. It depends on the polarization and on the complex structure. This gives two possible paths into the non-commutative world.

In [8] Nagao and Nakajima discovered Szendrői's non-commumtative DT invariants of the conifold [10]. By enlarging the perspective to consider moduli of complexes in the derived category and varying the stability condition one can move from moduli spaces of sheaves on the commutative resolution (giving the classical DT invariants) to moduli spaces of modules for the algebra of the non-commutative resolution (giving Szendrői's non-commumtative DT invariants). Moveover, there is a wall and chamber decomposition of the space of stability conditions with explicit wall crossing formulas for the invariants. The picture was enhanced to motivic invariants in [6]. Indeed Kontsevich and Soibelman define a *polarization* of a non-commutative variety to be this extra data of a stability condition [4].

The goal of the talk is to take the second path into non-commutative geometry by making a non-commutative deformation of the three dimensional algebra. We present the simple case of quantum $\mathbb{C}^3$.

To begin consider the more general example of three dimensional Calabi–Yau algebras $S_{a,b,c}$ defined as the quotient of $\mathbb{C}\langle x,y,z\rangle$ by the cyclic derivatives of $axyz + bxzy + c/3(x^3+y^3+z^3)$. In particular the algebra $S_{1,-q,0}$ defines quantum $\mathbb{C}^3$ and in the case $q=1$ we recover $\mathbb{C}[x,y,z]$.

Motivated by considerations in physics one expects that the space of all modules over the algebra $S_{a,b,c}$ should be described as the critical locus of a potential function on a bigger space of quantum fields. Indeed there exists a regular function $f_{a,b,c} : \mathrm{Hilb}(\mathbb{C}\langle x,y,z\rangle) \to \mathbb{C}$ from the smooth quasi-projective non-commutative Hilbert scheme whose critical locus $\{df_{a,b,c} = 0\}$ equals $\mathrm{Hilb}(S_{a,b,c})$, i.e. the scheme parameterizing finite dimensional cyclic left $S_{a,b,c}$ modules. We denote by $\mathrm{Hilb}^n(S_{a,b,c})$ the component pararmetrizing length n modules.

The virtual motive $[\mathrm{Hilb}^n(S_{a,b,c})]_{vir} := \mathbb{L}^{\dim(\mathrm{Hilb}^n(\mathbb{C}\langle x,y,z\rangle))/2}[\phi_{f_{a,b,c}}]$ is defined in terms of the *motivic class of vanishing cycles* $[\phi_{f_{a,b,c}}]$: a motivic class in the Grothendieck ring of varieties with the same underlying monodromic mixed Hodge structure as the usual sheaf of vanishing cycles [3]. We collect all the virtual motives in a partition function $Z_{a,b,c}(y) := \sum_{n\geq 0}[\mathrm{Hilb}^n(S_{a,b,c})]_{vir}y^n$.

For the calculation we need the following description of our potential function $f_{a,b,c}$. The smooth moduli space $\mathrm{Hilb}^n(\mathbb{C}\langle x,y,z\rangle)$ is a free $\mathrm{GL}(\mathbb{C}^n)$ quotient of the set of stable points in the linear space $V_n = \mathbb{C}^n \times \mathrm{End}(\mathbb{C}^n)^3$ [9]. Here the vector gives the cyclic generator and the three endomorphisms describe the action

[1]Elsewhere the motivic invariants are seen to have representation theoretic meaning with relation to co-homological Hall algebras and in string theory to BPS states [5].

of the co-ordinate functions. Now we can lift $f_{a,b,c}$ to a $\mathrm{GL}(\mathbb{C}^n)$ invariant cubic polynomial $F_{a,b,c}$ on V_n given by

$$F_{a,b,c}(v,X,Y,Z) = \mathrm{Tr}\left(aXYZ + bXZY + c/3(X^3+Y^3+Z^3)\right).$$

The motivic class of vanishing cycles for such a quasi-homogenous polynomial is much simpler and equals $[\phi_{F_{a,b,c}}] = [F^{-1}_{a,b,c}(1)] - [F^{-1}_{a,b,c}(0)]$ with a cyclic μ_3 action on the fiber over one. In this case it is shown [1] that the virtual motives $[\mathrm{Hilb}^n(S_{a,b,c})]$ can be recovered from those in the pre-quotient. Precisely it is shown that by setting

$$\mathcal{Z}_{a,b,c}(y) := \sum_{n\geq 0} \frac{[\phi_{F_{a,b,c}}]\cdot \mathbb{L}^{-n^2}}{[\mathrm{GL}(\mathbb{C}^n)]} y^n$$

we can recover $Z_{a,b,c}(y) = \mathcal{Z}_{a,b,c}(\mathbb{L}^{\frac{1}{2}}y)/\mathcal{Z}_{a,b,c}(\mathbb{L}^{-\frac{1}{2}}y)$. When $c=0$ the monodromy action is trivial and we can go further. Via a stratification of vanishing cycles we are left to compute the motivic classes of the moduli stacks of length n torsion sheaves on quantum $\mathbb{C}^2$

$$\mathcal{Z}_{a,b,0}(y) = \sum_{n\geq 0} \frac{[C_{-b/a}(n)]}{[\mathrm{GL}(\mathbb{C}^n)]} y^n$$

where $C_q(n) = \{(A,B) \in \mathrm{End}(\mathbb{C}^n)^2 : AB = qBA\}$ is the deformed commuting variety. To state the theorem we use the plethystic exponential Exp with the basic properties that $\mathrm{Exp}\,(u+v) = \mathrm{Exp}\,(u)\cdot \mathrm{Exp}\,(v)$ and $\mathrm{Exp}\left(\mathbb{L}^i y^j\right) = (1-\mathbb{L}^i y^j)^{-1}$.

Theorem. *Let $q \in \mathbb{C}^*$, then for q a primitive rth root of unity we have*

$$\mathcal{Z}_{1,-q,0}(y) = \mathrm{Exp}\left(\frac{2\mathbb{L}-1}{\mathbb{L}-1}\frac{y}{1-y} + (\mathbb{L}-1)\frac{y^r}{1-y^r}\right),$$

otherwise

$$\mathcal{Z}_{1,-q,0}(y) = \mathrm{Exp}\left(\frac{2\mathbb{L}-1}{\mathbb{L}-1}\frac{y}{1-y}\right).$$

To prove the result we make a stratification of the deformed commuting variety. In the generic case all torsion sheaves must be supported on the union of the two coordinate axis. The exponent $(2\mathbb{L}-1)/(\mathbb{L}-1)$ corresponds geometrically to the motivic class of the stack of such simple modules. In the rth root of unity case there can be r-dimensional simple modules supported away from the coordinate axis.

In forthcoming work [7] we will consider other non-commutative deformations of crepant resolutions.

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Higgs Bundles and Quiver Representations

Sergey Mozgovoy
(joint work with Markus Reineke)

We discussed the problem of counting semistable Higgs bundles (and twisted Higgs bundles) over a curve. So far this counting was done only for rank 2 (Hitchin) and 3 (Gothen) non-twisted Higgs bundles and for ranks $2-5$ co-Higgs bundles on $\mathbb{P}^1$ (twist of degree 2, Rayan). There exists a conjecture for the invariants of moduli spaces of arbitrary rank non-twisted Higgd bundles due to Hausel and Rodriguez-Villegas (generalized for twisted Higgs bunldes by myself). The goal of the talk was to explain the general strategy of counting semistable Higgs bundles, give an explicit formula for the invariants of moduli spaces of twisted Higgs bundles on $\mathbb{P}^1$ and to relate them to the invariants of moduli spaces of representations of some infinite symmetric quiver. We can verify on a computer for any rank and any twist that our formula gives the same result as a Hausel-Rodriguez-Villegas conjecture but could not prove that the formulas are equivalent

Morse theory of $\mathcal{D}$-module categories

Thomas Nevins
(joint work with Gwyn Bellamy, Chris Dodd, Kevin McGerty)

Suppose X is a smooth complex algebraic variety with the action of a reductive algebraic group G. In many instances, the Morse theory of X implies strong consequences for the ordinary or equivariant cohomology of X. I explained joint work with Bellamy, Dodd, and McGerty that develops parallel structure "one categorical level higher," for categories of equivariant $\mathcal{D}$-modules on X, with consequences for representation theory and topology.

Associated to X, G, and an additional choice of G-equivariant line bundle on T^*X, one typically gets an associated *Kirwan-Ness stratification* of T^*X. Such a stratification gives a filtration of the category $\mathcal{D}(X/G)$ of G-equivariant $\mathcal{D}$-modules on X (really, an appropriate dg enhancement of a suitable derived category of such modules) by singular support, or microsupport, in T^*X. In joint work with McGerty, we prove that this filtration realizes $\mathcal{D}(X/G)$ as glued from the categories microsupported on the strata: for each closed inclusion of strata there is a corresponding *recollement* of categories. In particular, we establish the existence and good properties of certain non-obvious adjoint functors. As a consequence, we prove a version of "hyperkaehler Kirwan surjectivity" for cotangent bundles.

Restricting attention to the top Kirwan-Ness stratum, i.e., the GIT-semistable locus, the resulting category is frequently identified with a category of deformation-quantization (DQ) modules on a smooth symplectic algebraic variety with a $\mathbb{G}_m$-action that rescales the symplectic form. In joint work with Bellamy, Dodd, and McGerty, we show that such a category satisfies an analogue of Kashiwara's equivalence for $\mathcal{D}$-modules: modules with certain support conditions are equivalent to modules over a smaller variety. This provides a "categorical cell decomposition" of such categories in many examples, leading to concrete applications to K-theory and Hochschild and cyclic homology of the categories, consequences for compact generation, and more. The resulting applications for linear invariants, specialized to particular examples, imply numerical and linear-algebraic assertions about, for example, categories of representations of wreath product symplectic reflection algebras.

The codimension-three conjecture for holonomic DQ-modules

François Petit

This talk is concerned with the codimension-three conjecture for holonomic Deformation Quantization modules (DQ-modules). This is an analogue of the codimension-three conjecture for microdifferential modules formulated by M. Kashiwara at the end of the 1970's and recently proved by M. Kashiwara and K. Vilonen (see [4]). The codimension-three conjecture is concerned with the extension of a holonomic microdifferential system over an analytic subset of the cotangent bundle of a complex manifold. Since DQ-modules provide a generalization of microdifferential modules to arbitrary symplectic manifolds (see [3]), it is natural to try to extend the codimension-three conjecture to holonomic DQ-modules. More precisely, we have obtained, in the case of DQ-modules, the following results.

Theorem 1. *Let X be a complex manifold endowed with a DQ-algebroid stack $\mathcal{A}_X$ such that the associated Poisson structure is symplectic. Let Λ be a closed Lagrangian analytic subset of X and Y be a closed analytic subset of Λ such that* $\operatorname{codim}_\Lambda Y \geq 3$. *Let $\mathcal{M}$ be a holonomic $(\mathcal{A}_X^{loc}|_{X\setminus Y})$-module, whose support is contained in $\Lambda\setminus Y$. Assume that $\mathcal{M}$ has an $\mathcal{A}_X|_{X\setminus Y}$-lattice. Then, $\mathcal{M}$ extends uniquely to a holonomic module defined on X whose support is contained in Λ.*

and

Theorem 2. *Let X be a complex manifold endowed with a DQ-algebroid stack $\mathcal{A}_X$ such that the associated Poisson structure is symplectic. Let Λ be a closed Lagrangian analytic subset of X and Y be a closed analytic subset of Λ such that* $\mathrm{codim}_\Lambda Y \geq 2$*. Let $\mathcal{M}$ be a holonomic $\mathcal{A}_X^{loc}$-module whose support is contained in Λ and let $\mathcal{M}_1$ be an $\mathcal{A}_X^{loc}|_{X \setminus Y}$-submodule of $\mathcal{M}|_{X \setminus Y}$. Then, $\mathcal{M}_1$ extends uniquely to a holonomic $\mathcal{A}_X^{loc}$-submodule of $\mathcal{M}$.*

We hope that the submodule version of the codimension-three conjecture for holonomic DQ-modules will find application to the representation theory of Cherednik algebras via the localization results due to M. Kashiwara and R. Rouquier [2].

Though the codimension-three conjecture for holonomic DQ-modules is strictly more general than the version for microdifferential modules, it relies deeply on the tools and ideas elaborated by M. Kashiwara and K. Vilonen. Indeed, our proof follows their general strategy. One of the main differences between the two problems is that we are working in a non-conical setting and thus, we have to adapt their strategy to such a framework.

We briefly describe the proof of the above statements. We keep the notation of Theorem 1 and 2 and denote by $j : X \setminus Y \to X$ the open embedding of $X \setminus Y$ into X, Then, the problem amount essentially to show that $j_*\mathcal{M}$ and $j_*\mathcal{M}_1$ are coherent. To prove this, we first notice this is a local problem. Thus, we can adapt a standard technique in complex analysis in several variables due to [5] and show that the coherence of $j_*\mathcal{M}$ and $j_*\mathcal{M}_1$ is equivalent to the coherence of the pushforward of these modules by a certain projection, the restriction of which to the support of the module is a finite map. Then, the coherence of the pushforward of $j_*\mathcal{M}$ and $j_*\mathcal{M}_1$ by the aforementioned projection follows from the below result due to M. Kashiwara and K. Vilonen. This is a difficult extension to coherent sheaves on $\mathcal{O}_X[[\hbar]]$ of a classical theorem due to Frisch-Guenot, Trautmann and Siu (see [1, 6, 7]).

Theorem 3 ([4, Theorem 1.6])**.** *Let X be a complex manifold and Y be a subvariety of X and $j : X \setminus Y \to X$ the open embedding of $X \setminus Y$ into X. If $\mathcal{N}$ is a coherent reflexive $\mathcal{O}_{X \setminus Y}[[\hbar]]$-module and* $\mathrm{codim} Y \geq 3$ *then $j_*\mathcal{N}$ is a coherent $\mathcal{O}_X[[\hbar]]$-module.*

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Frobenius Ext-algebras arising from Artin-Schelter Gorenstein Algebras

Daniel Rogalski
(joint work with Manuel Reyes, James Zhang)

Let k be an algebraically closed field. An Artin-Schelter (AS) Gorenstein algebra is a finitely generated $\mathbb{N}$-graded k-algebra $A = k \oplus A_1 \oplus A_2 \oplus \dots$ such that (i) A has finite injective dimension d as a right A-module, and (ii) $\operatorname{Ext}^i_A(k, A) = 0$ for $i \neq d$, and $\operatorname{Ext}^d_A(k, A) \cong k(\ell)$ (a graded shift of k), where $k = A/A_{\geq 1}$ is the trivial module. If (i) is replaced by the stronger condition that A has finite (graded) global dimension d, then A is called AS regular. Any AS Gorenstein algebra A has a graded automorphism $\mu_A : A \to A$ associated to it called the *Nakayama automorphism*, which is a natural generalization to higher dimension of the usual Nakayama automorphism which is defined when $d = 0$ and A is graded Frobenius. There is also the *homological determinant* $\operatorname{hdet} : \operatorname{Aut}_{gr}(A) \to k$, where $\operatorname{Aut}_{gr}(A)$ is the group of graded automorphisms of A, which is important in noncommutative invariant theory.

We conjecture that for any AS Gorenstein algebra A, $\operatorname{hdet} \mu_A = 1$ always holds. In a previous paper [1] this conjecture was shown to have numerous applications; for example, if the conjecture is true, one obtains that any AS Gorenstein algebra A with $\ell \neq 0$ is a graded twist of an AS Gorenstein algebra A' such that $\mu_{A'}$ has finite order [1, Theorem 7.8].

In the current work we describe a method that can be used to prove the conjecture in quite wide generality. Let $D_\epsilon(A)$ be the full triangulated subcategory of the derived category of graded left A-modules which consists of perfect complexes (bounded complexes of finitely generated projectives) with finite dimensional cohomologies. We show that this category has a Serre functor of the form $\Sigma^d T^\ell \Phi$, where Σ is the shift of complexes, T is induced by the shift of grading on modules, and Φ is induced by the action of the Nakayama automorphism μ_A on modules. Then given any object X in $D_\epsilon(A)$ which satisfies $X \cong \Phi(X)$ we show that the associated graded Ext algebra $E = \bigoplus_{i,j} \operatorname{Hom}_{D_\epsilon(A)}(X, \Sigma^i T^j(X))$ is Frobenius, and we give a formula for its Nakayama automorphism μ_E. As long as such an object X exists, by studying the relation between μ_E and μ_A and using that $\operatorname{hdet} \mu_E = 1$ (which is relatively easy to prove), we can prove the conjecture that $\operatorname{hdet} \mu_A = 1$. The study of these Ext algebras E and their Nakayama automorphisms μ_E seems interesting more generally, and we give other applications as well.

Ultimately, using this method we can prove the conjecture $\operatorname{hdet} \mu_A = 1$ for any AS Gorenstein algebra A such that $D_\epsilon(A) \neq 0$, by reducing to a case where a suitable object X exists. We show that the condition $D_\epsilon(A) \neq 0$ holds in most important cases, for example whenever A is a factor ring of an AS regular algebra, and we conjecture that it holds for all AS Gorenstein algebras.

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Categorification of $\mathfrak{gl}(1|1)$–representations

Antonio Sartori

The representation theory of the complex general Lie superalgebra $\mathfrak{gl}(1|1)$, although quite elementary, presents interesting combinatorial properties. For example, the quantum invariant of links corresponding to its vector representation $\mathbb{C}^{1|1}$ is the Alexander polynomial. This motivates our interest in computing intertwining operators between tensor products of representations and categorifying them.

In the talk, we describe in full detail the category of weight representations. Contrary to the case of usual semisimple Lie algebras, finite-dimensional representations of $\mathfrak{gl}(1|1)$ need not be semisimple. In particular, the category of finite-dimensional weight representations decomposes into a semisimple and a non-semisimple summand.

We focus then on the semisimple monoidal subcategory generated by exterior powers of the natural representation: here we can define and compute an analog of Lusztig's canonical basis, and prove that it has integrality and positivity properties. We can then construct a categorification using subquotient categories of the BGG category $\mathcal{O}(\mathfrak{gl}(n))$.

Finally, we show how these categories have a natural geometric interpretation related to the geometry of the Springer fiber of hook type. In particular, the endomorphism rings of indecomposable projective modules in these categories are naturally isomorphic to the cohomology rings of attracting subvarieties for an action of a one-dimensional torus.

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Poisson and Hochschild-de Rham Homology adn Symplectic Resolutions

Travis Schedler
(joint work with Pavel Etingof)

I defined the notion of Hochschild-de Rham homology of a deformation quantization of a Poisson variety. In previous work, we defined Poisson-de Rham homology of a Poisson variety, which recovers in the smooth symplectic case the de Rham cohomology of the variety; more generally, for varieties admitting a smooth symplectic resolution, this conjecturally recovers the de Rham cohomology of the symplectic resolution. For affine varieties the degree zero Poisson-de Rham homology coincides with the usual Poisson homology. Similarly, the Hochschild-de Rham homology recovers the de Rham cohomology when quantizing smooth symplectic varieties and conjecturally varieties admitting symplectic resolutions, and the degree zero Hochschild-de Rham homology of an affine variety is its degree zero Hochschild homology. In the case that the variety has finitely many leaves, the Hochschild and Poisson-de Rham homologies are finite-dimensional, and zero in degree of absolute value greater than the dimension.

Some enhancements of derived categories of coherent sheaves and applications

Olaf M. Schnürer
(joint work with Valery A. Lunts)

Let X be a quasi-projective scheme over a field k. The unbounded derived category $D(\mathrm{Qcoh}(X))$ of quasi-coherent sheaves on X contains the bounded derived category $D^b(\mathrm{Coh}(X))$ of coherent sheaves and the category of perfect complexes $\mathfrak{Perf}(X)$ as full triangulated subcategories,

$$\mathfrak{Perf}(X) \subset D^b(\mathrm{Coh}(X)) \subset D(\mathrm{Qcoh}(X)).$$

The objects of $\mathfrak{Perf}(X)$ can be characterized as the compact objects of $D(\mathrm{Qcoh}(X))$.

Recall that an enhancement of a triangulated (k-)category $\mathcal{T}$ is a pretriangulated dg (k-)category $\mathcal{A}$ together with an equivalence $[\mathcal{A}] \cong \mathcal{T}$ where $[\mathcal{A}]$ is the homotopy category of $\mathcal{A}$. For example, the full dg subcategory $C(\mathrm{Qcoh}(X))_{\text{h-inj}}$ of h-injective objects of the dg category $C(\mathrm{Qcoh}(X))$ of complexes of quasi-coherent sheaves forms an enhancement of $D(\mathrm{Qcoh}(X))$. The obvious full dg subcategories of $C(\mathrm{Qcoh}(X))_{\text{h-inj}}$ define enhancements of $D^b(\mathrm{Coh}(X))$ and $\mathfrak{Perf}(X)$.

Fix a finite affine open covering $(U_i)_{i=1}^n$ of X and consider for any complex P of vector bundles on X its $*$-Čech resolution

$$\mathcal{C}_*(P) = \Big(\prod_i {}_{U_i}P \to \prod_{i<j} {}_{U_{ij}}P \to \dots \Big)$$

where ${}_UP = u_*u^*P$ for $u\colon U \hookrightarrow X$. If P is a bounded complex of vector bundles the complex $\mathcal{C}_*(P)$ is defined as the obvious totalization.

For integral X and bounded complexes P and Q of vector bundles the canonical map

$$\mathrm{Hom}_{[C(\mathrm{Qcoh}(X))]}(\mathcal{C}_*(P), \mathcal{C}_*(Q)) \to \mathrm{Hom}_{D(\mathrm{Qcoh}(X))}(\mathcal{C}_*(P), \mathcal{C}_*(Q))$$

is an isomorphism and hence the full dg subcategory of $C(\mathrm{Qcoh}(X))$ consisting of $*$-Čech resolutions $\mathcal{C}_*(P)$ of bounded complexes P of vector bundles forms an enhancement of $\mathfrak{Perf}(X)$. For smooth projective X this enhancement was considered in [BLL04].

We found a modification of this construction which provides an enhancement of $\mathfrak{Perf}(X)$ for any quasi-projective scheme X over k. The objects of this enhancement are bounded complexes of vector bundles and the morphism spaces are suitably defined dg submodules of $\mathrm{Hom}_{C(\mathrm{Qcoh}(X))}(\mathcal{C}_*(P), \mathcal{C}_*(Q))$. Using !-Čech resolutions we similarly define enhancements of $\mathfrak{Perf}(X)$ and $D^b(\mathrm{Coh}(X))$. We call these enhancements Čech enhancements.

Recall that a dg category $\mathcal{A}$ is called k-smooth if the diagonal dg bimodule $\mathcal{A}$ is a compact object of the derived category $D(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}})$ of dg $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$-modules.

We say that $\mathfrak{Perf}(X)$ (resp. $D^b(\mathrm{Coh}(X))$) is smooth over k if its h-injective enhancement is k-smooth as a dg category.

Using Čech enhancements we prove the following three theorems.

Theorem 4 (Homological versus geometric smoothness)**.** *Let $\Delta\colon X \to X \times X$ be the diagonal immersion. The following three conditions are equivalent:*

(1) $\mathfrak{Perf}(X)$ *is smooth over* k;
(2) $\Delta_*(\mathcal{O}_X) \in \mathfrak{Perf}(X \times X)$;
(3) X *is smooth over* k.

Theorem 5. *If the field k is perfect then $D^b(\mathrm{Coh}(X))$ is smooth over k.*

Theorem 6 (Fourier-Mukai kernels and dg bimodules)**.** *Let X and Y be quasi-projective schemes over a field k and consider the projections $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$. Then there are dg algebras A and B and equivalences of triangulated categories*

$$\theta_X\colon D(\mathrm{Qcoh}(X)) \xrightarrow{\sim} D(A),$$
$$\theta_Y\colon D(\mathrm{Qcoh}(Y)) \xrightarrow{\sim} D(B),$$
$$\theta_{X\times Y}\colon D(\mathrm{Qcoh}(X \times Y)) \xrightarrow{\sim} D(A^{\mathrm{op}} \otimes B),$$

such that for any $K \in D(\mathrm{Qcoh}(X \times Y))$ with corresponding $M = \theta_{X\times Y}(K)$ the diagram

$$\begin{array}{ccc} D(\mathrm{Qcoh}(X)) & \xrightarrow{\mathbf{R}q_*(p^*(-)\otimes^{\mathbf{L}} K)} & D(\mathrm{Qcoh}(Y)) \\ {\scriptstyle \theta_X}\downarrow{\scriptstyle \sim} & & {\scriptstyle \theta_Y}\downarrow{\scriptstyle \sim} \\ D(A) & \xrightarrow{-\otimes^{\mathbf{L}}_A M} & D(B) \end{array}$$

commutes up to an isomorphism of triangulated functors.

Some variants of these results are claimed in the literature without proof or with gaps in the proofs, cf. the discussion in [LS14].

All three theorems admit short heuristic arguments making them plausible. However, turning these arguments into rigorous proofs seems to be hard. The different functors involved (inverse image, tensor product, direct image, **R**Hom) are usually computed via different types of replacements (h-flat, h-injective) which makes it difficult to treat them compatibly. It would be desirable to lift all these functors and the adjunctions among them from the derived level to the dg level of enhancements.

For simplicity we stated our results here for quasi-projective schemes over a field. This assumption can be weakened, see [LS14], where the above theorems and some more results characterizing properness are proved.

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Ring-theoretic blowing down

Susan J. Sierra

(joint work with Dan Rogalski, Toby Stafford)

This work is part of an ongoing project to extend tools from the algebraic geometry of projective surfaces to noncommutative graded domains of GK-dimension 3, with the ultimate goal of classifying such algebras.

We first discuss the algebraic geometry we wish to generalise. Fix an algebraically closed field k; all varieties and algebras will be defined over k. Let X be a smooth projective surface. Let $x \in X$, and let $\pi : Bl_x(X) \to X$ be the blowup of X at x; recall that π is a *monoidal transformation*. It is well-known that:

Proposition 5. *$Bl_x(X)$ is a smooth projective surface. If $L = \pi^{-1}(x)$ is the exceptional locus of π, then $L \cong \mathbb{P}^1$ and $L.L = -1$.*

In the proposition above, π contracts L to x by construction. A celebrated theorem ("Castelnuovo's contraction criterion") says that the converse also holds: the properties of L characterise curves that can be contracted to smooth points.

Theorem 7 (Castelnuovo)**.** *Let Y be a smooth projective surface, and let L be a curve on Y so that $L \cong \mathbb{P}^1$ and $L.L = -1$. Then there is a smooth projective surface X and a birational morphism $\pi : Y \to X$ so that L is the exceptional locus of π and $Y \cong Bl_x(X)$, where $x = \pi(L)$.*

The other crucial theorem in the subject is the following consequence of Zariski's Main Theorem, which shows that monoidal transformations are the building blocks of birational geometry of surfaces.

Theorem 8 (Zariski)**.** *Let $X \dashrightarrow Y$ be a birational map of smooth projective surfaces. Then there are a smooth projective surface Z and compositions of monoidal transformations $Z \to X$, $Z \to Y$ so that*

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & \dashrightarrow & Y \end{array}$$

commutes.

The fundamental example of this theorem is the isomorphism between the blowup of $\mathbb{P}^2$ at two points and the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point. The diagram above becomes:

$$\begin{array}{ccc} & Bl_r(\mathbb{P}^1 \times \mathbb{P}^1) \cong Bl_{p,q}\mathbb{P}^2 & \\ \swarrow & & \searrow \\ \mathbb{P}^1 \times \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^2 \end{array} \tag{1}$$

To generalise this to the noncommutative setting, we first must extend some definitions. Let R, S be connected graded k-algebras that are domains. We say that R, S are *birational* if $Q_{gr}(R)_0 \cong Q_{gr}(S)_0$. (If R is a homogeneous coordinate ring of a projective variety X, then $Q_{gr}(R)_0 \cong k(X)$, so this generalises the definition from commutative algebraic geometry.) For example, let S be a quadratic Sklyanin algebra (a "noncommutative $\mathbb{P}^2$) and let S' be a cubic Sklyanin algebra (or more generally a smooth noncommutative quadric surface). Michel Van den Bergh [2, Theorem 13.4.1] has proved that S and S' are birational.

Our results apply to *elliptic algebras*: an elliptic algebra is a connected graded domain R containing a central $g \in R_1$ so that $R/(g)$ is isomorphic to a twisted homogeneous coordinate ring on an elliptic curve E. (We say that E is *associated to* R.) For example, the 3-Veronese $T := S^{(3)}$ of S is elliptic, as is $T' := (S')^{(4)}$.

For elliptic algebras, there is a good analogue of a monoidal transformation. Let R be an elliptic algebra with $\dim R_1 \geq 4$. Then $R_1/(g)$ may be identified with global sections of an invertible sheaf $\mathcal{L}$ on E. If $p \in E$, we define $Bl_p(R)$ to be the subalgebra of R generated by the elements of R_1 whose images mod g vanish at p. (The algebra $Bl_p(R)$ also occurs in [3] as the coordinate ring of the more categorical blowups defined there.) A theorem of Rogalski [1, Lemma 9.1] says that $Bl_p(R)$ has a line module L so that $R/Bl_p(R) \cong L^{\oplus \mathbb{Z}}$ as an (ungraded) $Bl_p(R)$-module. We refer to L as the *exceptional line module* for $Bl_p(R)$.

Let R be an elliptic algebra. A graded R-module L is a *line module* if L is cyclic and has Hilbert series $1/(1-s)$. We say that L has *self-intersection* (-1) if $Ext^1_R(L, L) = 0$ (there is also an equivalent definition using noncommutative intersection theory). Then we prove noncommutative versions of Proposition 5 and Theorem 7:

Proposition 6. *Let R be an elliptic algebra with associated elliptic curve E, let $p \in E$, and let L be the exceptional line module for $R' := Bl_p(R)$. If the image of L has projective dimension 1 over $R'/(g-1)$, then L has self-intersection (-1).*

(We note that it is possible for the condition on the projective dimension of L to fail and for $Ext^1_R(L, L)$ to be nonzero, so the analogy with commutative geometry is not exact.)

Theorem 9. *Let R be an elliptic algebra with associated elliptic curve E and let L be a line module with self-intersection (-1). Then there is an elliptic algebra R' associated to E and a point $p \in E$ so that $R \cong Bl_p(R')$, with exceptional line L.*

As yet, there is no general analogue of Theorem 8. We do prove, however:

Theorem 10. *Let E be the elliptic curve associated to T', defined above, and let $r \in E$ be generic. Then there is a quadratic Sklyanin algebra S so that $T = S^{(3)}$ is associated to E and points $p, q \in E$ so that*

$$Bl_r(T') \cong Bl_{p,q}(T).$$

Our theorem also holds for smooth quadric surfaces, giving a noncommutative version of (1).

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Serre functors and derived equivalences for hereditary Artin algebras

Adam-Christiaan van Roosmalen

(joint work with Donald Stanley)

I reported on recent research by Donald Stanley and myself in which we studied a connection between Serre functors and derived equivalences (see [3]). Throughout, let k be a field. We will assume that all categories are k-linear. For a triangulated category $\mathcal{C}$, we will write Σ for the suspension.

1. Serre duality

Let $\mathcal{C}$ be a Hom-finite triangulated category. A *Serre functor* ([2]) is a triangle autoequivalence $\mathbb{S} : \mathcal{C} \to \mathcal{C}$ together with isomorphisms

$$\eta_{A,B} : \text{Hom}(A, B) \cong \text{Hom}(B, \mathbb{S}A)^*,$$

for any $A, B \in \mathcal{C}$, which are natural in A, B and where $(-)^*$ is the vector space dual.

We say that an abelian category $\mathcal{A}$ has *Serre duality* if $D^b\mathcal{A}$ admits a Serre functor. It has been shown in [2] that the following categories have Serre duality:

- the category $\operatorname{mod}\Lambda$ of finite-dimensional modules over a finite-dimensional algebra Λ with finite global dimension, and
- the category $\operatorname{Coh}\mathbb{X}$ of coherent sheaves on a smooth projective variety $\mathbb{X}$.

2. t-structures

Let $\mathcal{C}$ be a triangulated category. A t-structure ([1]) on $\mathcal{C}$ is a pair $(\mathcal{U}, \mathcal{V})$ of full subcategories of $\mathcal{C}$ satisfying the following conditions:

(1) $\Sigma\mathcal{U} \subseteq \mathcal{U}$ and $\mathcal{V} \subseteq \Sigma\mathcal{V}$,
(2) $\operatorname{Hom}(\Sigma\mathcal{U}, \mathcal{V}) = 0$,
(3) $\forall C \in \mathcal{C}$, there is a triangle $U \to C \to V \to \Sigma U$ with $U \in \mathcal{U}$ and $V \in \Sigma^{-1}\mathcal{V}$.

Furthermore, we will say that the t-structure $(\mathcal{U}, \mathcal{V})$ is *bounded* if

$$\bigcup_{n\in\mathbb{Z}} \Sigma^n\mathcal{U} = \bigcup_{n\in\mathbb{Z}} \Sigma^n\mathcal{V} = \mathcal{C}.$$

The *heart* of a t-structure $(\mathcal{U}, \mathcal{V})$ is defined to be $\mathcal{U} \cap \mathcal{V}$. It has been shown in [1, Théorème 1.3.6] that the heart is an abelian category.

When $\mathcal{C}$ is the bounded derived category $D^b\mathcal{A}$ of an abelian category $\mathcal{A}$, the embedding $\mathcal{H} \to D^b\mathcal{A}$ lifts to a triangle functor $D^b\mathcal{H} \to D^b\mathcal{A}$, called the *realization functor*. In general, the realization functor is not an equivalence. The following theorem is standard (see [1, Proposition 3.1.16]).

Theorem 11. *Let $\mathcal{A}$ be an abelian category, and let $(\mathcal{U}, \mathcal{V})$ be a t-structure in $D^b\mathcal{A}$ with heart $\mathcal{H}$. The realization functor $D^b\mathcal{H} \to D^b\mathcal{A}$ is an equivalence if and only if for all $A, B \in \mathcal{H}$, all $n \geq 2$, and every morphism $f : A \to \Sigma^n B$, there is a monomorphism $B \to C$ in $\mathcal{H}$ such that $A \to \Sigma^n B \to \Sigma^n C$ is zero.*

3. Main result

A first connection between Serre functors and derived equivalences is given in the following proposition.

Proposition. *Let $\mathcal{A}$ be an abelian category, and let $(\mathcal{U}, \mathcal{V})$ be a t-structure in $D^b\mathcal{A}$ with heart $\mathcal{H}$. If the realization functor $D^b\mathcal{H} \to D^b\mathcal{A}$ is an equivalence, then $(\mathcal{U}, \mathcal{V})$ is bounded and $\mathbb{S}\mathcal{U} \subseteq \mathcal{U}$.*

One can now wonder when the converse of the previous proposition holds.

Question. *Let $\mathcal{A}$ be an abelian category of which the bounded derived category $D^b\mathcal{A}$ admits a Serre functor $\mathbb{S} : D^b\mathcal{A} \to D^b\mathcal{A}$, and let $(\mathcal{U}, \mathcal{V})$ be a t-structure on $D^b\mathcal{A}$ with heart $\mathcal{H} = \mathcal{U} \cap \mathcal{V}$. For which categories $\mathcal{A}$ (and possibly for which restricted class of t-structures $(\mathcal{U}, \mathcal{V})$ on $D^b\mathcal{A}$) are the following statements equivalent:*

(1) *the realization functor $D^b\mathcal{H} \to D^b\mathcal{A}$ is an equivalence,*
(2) *the t-structure $(\mathcal{U}, \mathcal{V})$ is bounded and $\mathbb{S}\mathcal{U} \subseteq \mathcal{U}$?*

The main result I reported on, is that the answer to the previous question is positive when $\mathcal{A}$ is the module category of a finite-dimensional hereditary algebra:

Theorem 12. *Let* Λ *be a finite-dimensional hereditary algebra and write* $\operatorname{mod}\Lambda$ *for the category of finite-dimensional right* Λ*-modules. Let* $(\mathcal{U}, \mathcal{V})$ *be a t-structure in* $D^b \operatorname{mod}\Lambda$ *with heart* $\mathcal{H}$.

The realization functor $D^b\mathcal{H} \to D^b\mathcal{A}$ *is an equivalence if and only if* $(\mathcal{U}, \mathcal{V})$ *is bounded and* $\mathbb{S}\mathcal{U} \subseteq \mathcal{U}$.

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On Quantum Symmetry

Chelsea Walton

Let k be an algebraically closed field of characteristic zero. The purpose of this work is to study quantum analogues of group actions on commutative k-algebras. Here, we restrict our attention to the actions of finite quantum groups, i.e. finite dimensional Hopf algebras H. Two important subclasses of such H are those that are *semisimple* (that is to say, semisimple as an algebra; these Hopf algebras are always finite dimensional) and those that are *pointed* (that is to say, all simple H-comodules are 1-dimensional).

Naturally there are many choices on what (quantum) k-algebras we can act on, but we are first motivated from the viewpoint of classic invariant theory and algebraic geometry, where the examination of Hopf actions on commutative domains over k is of interest. The classification of semisimple Hopf actions on such k-algebras is completely understood by following result with Pavel Etingof.

Theorem 1. [2, Theorem 1.3] Any action of semisimple Hopf algebra H over k on a commutative domain over k factors through a finite group action.

The theorem fails if any of the hypotheses are omitted: see [2, Remark 4.3] for 'domain' being omitted, [2, Example 5.10] and [5, Examples 7.4–7.6] for 'commutative' being omitted, and Theorem 2 below for the omission of semisimplicity.

Next, Etingof and I investigate finite dimensional Hopf actions on quantizations of commutative domains. An important case of this task was established when the module algebra was the n-th Weyl algebra $A_n(k)$ with standard filtration and the H-action preserves the filtration of $A_n(k)$. Here, we get that the H-action factors through a finite group action [2, Corollary 1.4]. Generalizing this result to actions on more general quantizations of commutative domains (including rings of differential operators, enveloping algebras of a finite dimensional Lie algebra, quantized quiver varieties, etc.) is a work in progress with Juan Cuadra [1]. Here, we will not assume that the H-action preserves the filtration of the module algebra.

In contrast to Theorem 1, we have that there are many actions of finite dimensional nonsemisimple Hopf algebras on commutative domains that do not factor through a group action. Etingof and I study the case when H is pointed of *finite Cartan type*. The study boils down to H-actions on fields (that do not factor through a group action), and we refer to such H as *Galois-theoretical*.

Theorem 2. [3, Theorem 1.2] There exist several examples and non-examples of Galois-theoretical Hopf algebras, including the Taft algebras, $u_q(\mathfrak{sl}_2)$, and some Drinfeld twists of other small quantum groups.

The classification of Galois-theoretical finite dimensional pointed Hopf algebras (of finite Cartan type) is a work in progress [4]. Please see the references below for several questions and conjectures related to this program.

References

[1] J. Cuadra, P. Etingof, and C. Walton, *Semisimple Hopf actions on quantizations* (preprint), in progress.

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Category $\mathcal{O}$ and KLR algebras

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We discuss a generalization of the BGG category $\mathcal{O}$ which arises from other symplectic varieties, in particular from quiver varieties. This approach works by defining a category of D-modules that category $\mathcal{O}$ is a quotient of. Using topological techniques, we can compute this category as the representations of a convolution algebra with a combinatorial description. In the quiver case, what arises is a generalization of KLR algebras, which we call weighted KLR algebras. We explain how these relations arise from the geometry of quivers.

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