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Functional topology for geometric settings

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ABSTRACT

In this paper we propose two approaches to connect the functional topology theory of Clementino et al. (2004) [4] with the situation in the category Sch of schemes. In a first approach, we extend an example from [4] to a more general pattern for obtaining a class of "closed morphisms" from two auxiliary classes, "closed embeddings" and newly added "surjections". In a second approach we apply [4] to the presheaf category over Sch, and recover the notions of properness and separatedness on Sch.

1. Introduction

Among the most fundamental properties in general topology are the Hausdorff property and compactness of topological spaces, and their relative counterparts, separatedness and properness for continuous functions. Efforts of abstracting these notions to more general categorical contexts endowed with some notion of "closedness" or "properness" date back at least to the seventies with work of Herrlich [12], Manes [19], Penon [23], and [22]. A richer theory was developed for categories endowed with a factorization system [1], see the work of Herrlich, Salicrup and Strecker [13] with applications in topology, group and order theory. In the nineties, once the notion of a "closure operator" on a category endowed with a factorization structure was available, results quite close to the original topological situation were obtained, see [5], with applications to Top, Birkhoff closure spaces, uniform spaces, topological groups and locales. In [25] Tholen observed that both the class of separated morphisms and the class of proper morphism can be expressed in terms of an auxiliary class of "closed morphisms" and these ideas grew out to a theory called "functional approach to topology" developed in the context of the auxiliary class of closed morphisms linked to a given factorization system; see the work of Clementino, Giuli and Tholen [4]. Applications of this setting include approach spaces, a common generalization of topological and metric spaces [6]. Recently, in order to capture more general categories of lax algebras, Hofmann and Tholen [14] adapted the setting, replacing the class of closed morphisms by an auxiliary class of "proper maps".

The present paper grew out of the observation of the striking similarities between the ingredients and results in [4] on the one hand, and the situation in algebraic geometry on the other hand. It is well known that the notions of separatedness and properness of underlying topological spaces are inappropriate for schemes. Indeed, spectra of commutative rings, the building blocks for schemes, are always compact and never (unless in trivial cases) Hausdorff, whereas one actually wants these affine schemes to be "morally" Hausdorff but not compact. The correct notions due to Grothendieck [24] – called separatedness and properness both for objects and morphisms – are based upon the familiar "correct" notion of closed morphisms (i.e. morphisms of schemes that are closed on the underlying topological level) by means of the same categorical definitions used in all higher approaches: separated means that the diagonal morphism is closed, proper (or rather the part which – in the scheme literature – is usually referred to as "universally closed"; in the scheme language the word proper is reserved for universally closed + separated + finite type) means pullback stably closed.

Another feature which raises hopes that the category of schemes may fit into the approach of [4] is the prominent role of a natural class of "closed embeddings". In the functional topology approach, these are obtained from a "background"

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factorization system $(\mathbb{E}, \mathbb{M})$ as the closed morphisms in $\mathbb{M}$. In Sch, we have the natural class of closed immersions. However, it is precisely here that lies the main obstacle: in the category of schemes, there is no natural notion of "not necessarily closed" image of a morphism, whence no background factorization system. Although the closed immersions do form part of a factorization system themselves, taking this as background does not fit into the theory of [4], and it certainly does not allow to distinguish closed morphisms based upon factorizations through closed embeddings in the way this is possible for topological spaces. In this paper we present several solutions which allow to naturally connect the two setups in spite of this difficulty.

This paper is a first step in a broader project where the aim is to investigate the role of functional topology in geometric settings, with the eye on possible extensions to settings in so called non-commutative geometry. Currently, notions of properness are often based upon cohomological properties (see e.g. [17,26]) rather than upon the internal structure of spaces (investigated for instance in [21]). With the growing realization that not all "spaces" can be modeled entirely on cohomological ground (see [18,15,16]), the development of a functional approach, and the comparison with existing cohomological approaches, seems worthwhile and remains work in progress.

The paper is divided into three main sections.

In Section 2, we present our account of what can be considered as "basic functional topology". If one agrees that the notions of separated and proper morphisms and their properties and interactions are central to the theory, one can try to develop things with as little additional data as possible. We discuss both the possibility of taking a single class $\mathbb{F}$ of "closed morphisms" as input, or the possibility of considering an additional $\mathbb{F}_0 \subseteq \mathbb{F}$ of "closed embeddings". Here, we do not include a notion of more general embeddings. To obtain some of the familiar behavior of separated and proper morphisms, we identify the crucial property that for any morphisms, the diagonal is closed if and only if it is proper if and only if it is a closed immersion. Our presentation is similar in spirit to [25], where separated morphisms are immediately defined with respect to the proper morphisms. Note that it is possible to take $\mathbb{F}$ equal to the proposed (pullback stable) class of proper morphisms in both approaches.

In Section 3 we work relative to a factorization system as in [4]. We propose a definition of the class of closed morphisms based upon two auxiliary classes (together called a "closed structure"), a class of "closed embeddings", required to be composition closed and pullback stable, and a class of "surjections", required to be composition closed. Our definition of the associated class of "closed morphisms" (Definition 3.12) generalizes Example 3.11 from [4]. Precisely, we declare a morphism to be closed if for every closed embedding landing in the domain, the factorization of the composition is given by a surjection followed by a closed embedding. We further collect some properties related to [4] and Section 2 in Section 3.3.

Our application to the category Sch is twofold. In a first step, in Section 3.6, we characterize within this formalism the morphisms whose natural closed image in the larger category LRS of locally ringed spaces is a scheme. More precisely, we describe these morphisms as the sch-closed morphisms with respect to a relative closure operator sch in the sense of [3].

In a second step, in Section 3.8, we choose on the category Sch the closed structure with the closed immersions as closed embeddings, and with as surjections the surjective (on the underlying set theoretic level) morphisms of schemes. If we restrict our attention to the sch-closed morphisms, among them the closed (on the underlying topological level) morphisms now correspond precisely to the morphisms that are closed with respect to the newly introduced closed structure. Further and most importantly, the closed structure yields the correct classes of proper and separated morphisms. This result is "lifted" in a suitable sense (see Section 3.7) from the situation on Top, endowed with the (surjections, closed embeddings) as closed structure relative to the (dense maps, closed embeddings) factorization system. The observation that this equivalently describes the standard closed morphisms on Top (see Section 3.4) is crucial to our approach.

In Section 4 we develop a general approach for capturing proper and separated morphisms based upon presheaf categories. Here, the idea is to use the standard factorization system on the presheaf category over an original category of interest as "background". This is a very nice factorization system which fits into the setup of [4]. Our starting point is a pullback closed class of morphisms in the original category, which we would like to consider as "proper morphisms". Using the notion of representable morphisms, we first define a class of morphisms on the presheaf category which restricts to the class of candidate proper morphisms on the original category. Then we introduce closed embeddings as being precisely the images of the morphisms in this class. In this way, we always obtain a closed structure (Section 4.3) on the presheaf category. Finally, in Section 4.4, we prove that in case we start with the category Sch – upon restriction to this category – this procedure yields the correct classes of separated and proper morphisms.

2. Basic functional topology

In [4], functional topology was introduced as a categorical framework in which certain ideas from topology, revolving around closed embeddings, closed morphisms, proper morphisms and separated morphisms, can be developed. The approach in [4] makes use of a "background" factorization system for this development. In particular, one makes use of a notion of "embeddings" – the second class in the factorization system – which is such that the closed embeddings are precisely the closed morphisms that are at the same time embeddings. In this section, we present a more basic approach which only takes closed morphisms (see Definition 2.7), and in a second version closed morphisms and closed embeddings (see Definition 2.10), as an input. We show that many of the standard properties relating the secondary notions of proper and separated morphisms hold true under the assumption that for the diagonal of an arbitrary morphism, it is equivalent to say that it is closed, or that it is proper, or that it is a closed embedding. This fundamental property is fulfilled in various different settings where natural classes of separated and proper morphisms are present and share a number of basic properties, like pullback stability and closure under compositions of the individual classes, and the fact that the separated morphisms are left cancellable with respect to the proper morphisms. Our setup encompasses both the setup of [4], and the situation in various categories of locally ringed spaces, like the category of schemes and the category of complex analytic spaces. As we will discuss in Sections 3 and 4, the theory of [4] does not apply to these categories without certain modifications either to the theory, or else to the categories. Our treatment in this section is similar in spirit to [25].

2.1. Classes of morphisms

Let $\mathcal{C}$ be a category. In this section we introduce some notation and terminology concerning classes of morphisms in $\mathcal{C}$.

We will make use of the standard classes Mor of all morphisms, Iso of isomorphisms, Mono of monomorphisms and Epi of epimorphisms. For two classes of morphisms $\mathbb{F}$ and $\mathbb{H}$, we denote by $\mathbb{F} \subseteq \mathbb{H}$ that $f \in \mathbb{F}$ implies $f \in \mathbb{H}$.

Definition 2.1. Let $\mathbb{F}$ be a class of morphisms.

(1) A morphism g is $\mathbb{F}$-*left cancellable* if $gh \in \mathbb{F}$ implies $h \in \mathbb{F}$.
(2) A morphism g is $\mathbb{F}$-*right cancellable* if $hg \in \mathbb{F}$ implies $h \in \mathbb{F}$.
(3) A morphism g is $\mathbb{F}$-*dense* if $g = fu$ with $f \in \mathbb{F}$ implies $f \in \mathsf{Iso}$.
(4) A morphism g is $\mathbb{F}$-*orthogonal* if for $fu = vg$ with $f \in \mathbb{F}$, there is a unique morphism w with $fw = v$ and $wg = u$.

We thus obtain the corresponding classes $\mathbb{F} - \mathsf{LCan}$ of $\mathbb{F}$-left cancellable morphisms, $\mathbb{F} - \mathsf{RCan}$ of $\mathbb{F}$-right cancellable morphisms, $\mathbb{F} - \mathsf{Dense}$ of $\mathbb{F}$-dense morphisms, $\mathbb{F} - \mathsf{Ortho}$ of $\mathbb{F}$-orthogonal morphisms.

We mention the following easy fact:

Lemma 2.2. *If $\mathbb{F}$ is pullback-stable, then* $\mathsf{Mono} \subseteq \mathbb{F} - \mathsf{LCan}$.

2.2. Proper and separated morphisms

Let $\mathcal{C}$ be a finitely complete category. For a morphism $g : X \longrightarrow Y$, the *diagonal* Δ_g of g is the unique morphism $\Delta_g = (1_X, 1_X) : X \longrightarrow X \times_Y X$ to the pullback $X \times_Y X$ of g along itself.

Definition 2.3. Let $\mathbb{F}$ be a class of morphisms.

(1) A morphism g is $\mathbb{F}$-*proper* if every pullback of g is in $\mathbb{F}$.
(2) A morphism g is $\mathbb{F}$-*separated* if the diagonal $\Delta_g \in \mathbb{F}$.
(3) A morphism g is $\mathbb{F}$-*perfect* if it is $\mathbb{F}$-proper and $\mathbb{F}$-separated.

We thus obtain the corresponding classes $\mathbb{F} - \mathsf{Prop}$ of $\mathbb{F}$-proper morphisms, $\mathbb{F} - \mathsf{Sep}$ of $\mathbb{F}$-separated morphisms and $\mathbb{F} - \mathsf{Perf}$ of $\mathbb{F}$-perfect morphisms.

Lemma 2.4. (1) $\mathbb{F} - \mathsf{Prop} \subseteq \mathbb{F}$.
(2) $\mathbb{F} - \mathsf{Prop}$ *is pullback-stable.*
(3) *If* $\mathbb{F} \subseteq \mathbb{G}$ *then* $\mathbb{F} - \mathsf{Sep} \subseteq \mathbb{G} - \mathsf{Sep}$ *and* $\mathbb{F} - \mathsf{Prop} \subseteq \mathbb{G} - \mathsf{Prop}$.
(4) *If* $\mathbb{F}_0 \subseteq \mathbb{F}$ *with* $\mathbb{F}_0$ *pullback-stable, then* $\mathbb{F}_0 \subseteq \mathbb{F} - \mathsf{Prop}$.
(5) *If* $\mathsf{Iso} \subseteq \mathbb{F}$, *then* $\mathsf{Iso} \subseteq \mathbb{F} - \mathsf{Prop}$.
(6) *If $\mathbb{F}$ is closed under compositions, then so is* $\mathbb{F} - \mathsf{Prop}$.
(7) $\mathsf{Mono} = \mathsf{Iso} - \mathsf{Sep}$.
(8) *If* $\mathsf{Iso} \subseteq \mathbb{F}$, *then* $\mathsf{Mono} \subseteq \mathbb{F} - \mathsf{Sep}$.

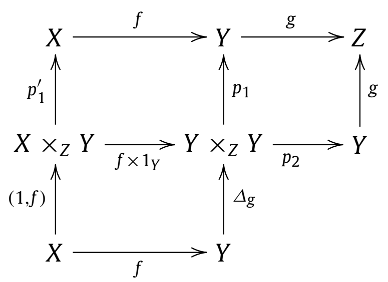
Lemma 2.5. *Suppose* $\mathbb{F} - \mathsf{Sep} = (\mathbb{F} - \mathsf{Prop}) - \mathsf{Sep}$. *Then we have:*

(1) $\mathbb{F} - \mathsf{Sep}$ *is pullback-stable.*
(2) *If $\mathbb{F}$ is closed under compositions, then so is* $\mathbb{F} - \mathsf{Sep}$.
(3) $(\mathbb{F} - \mathsf{Sep}) - \mathsf{LCan} = \mathsf{Mor}$.

We have the following key lemma:

Lemma 2.6. *If* $\mathsf{Iso} \subseteq \mathbb{F}$ *and if $\mathbb{F}$ is pullback-stable and closed under compositions, then* $\mathsf{Mono} \subseteq \mathbb{F} - \mathsf{Sep} \subseteq \mathbb{F} - \mathsf{LCan}$.

Proof. Consider $gf \in \mathbb{F}$ with $g \in \mathbb{F} - \mathsf{Sep}$. We obtain the following diagram of pullbacks:



Since $gf \in \mathbb{F}$, the pullback $p_2(f \times 1_Y) \in \mathbb{F}$. Since $\Delta_g \in \mathbb{F}$, the pullback $(1, f) \in \mathbb{F}$. Thus the composition $p_2(f \times 1_Y)(1, f) = p_2\Delta_g f = f \in \mathbb{F}$. $\square$

2.3. Closed classes and closed pairs

Let $\mathcal{C}$ be a finitely complete category.

Definition 2.7. A *pre-closed class* on $\mathcal{C}$ is a class of morphisms $\mathbb{F}$ with $\mathsf{Iso} \subseteq \mathbb{F}$ and such that $\mathbb{F}$ is closed under composition. Morphisms in $\mathbb{F}$ are called *closed morphisms*. If a pre-closed class $\mathbb{F}$ on $\mathcal{C}$ is chosen, we put $\mathsf{Prop} = \mathbb{F} - \mathsf{Prop}$, $\mathsf{Sep} = \mathbb{F} - \mathsf{Sep}$ and $\mathsf{Perf} = \mathbb{F} - \mathsf{Perf}$ and we simply speak of *proper*, *separated* and *perfect* morphisms. A pre-closed class $\mathbb{F}$ is called a *closed class* if

(c) $\mathsf{Sep} \subseteq \mathsf{Prop} - \mathsf{Sep}$.

A pre-closed class $\mathbb{F}$ is called a *stable class* if $\mathbb{F}$ is pullback-stable.

Clearly, a stable class is a closed class.
We obtain the following properties:

Proposition 2.8. *Let $\mathbb{F}$ be a pre-closed class on $\mathcal{C}$.*

(1) $\mathsf{Iso} \subseteq \mathsf{Perf}$.
(2) $\mathsf{Mono} \subseteq \mathsf{Sep}$.
(3) Prop *is closed under composition and is pullback-stable.*
(4) *Let $\mathbb{P}$ be closed under composition and pullback-stable with* $\mathsf{Sep} \subseteq \mathbb{P} - \mathsf{Sep}$. *Then* $\mathsf{Sep} \subseteq \mathbb{P} - \mathsf{LCan}$.

Let $\mathbb{F}$ be a closed class on $\mathcal{C}$.

(1) $\mathsf{Sep} = \mathsf{Prop} - \mathsf{Sep}$.
(2) Prop, Sep *and* Perf *are closed under composition and are pullback-stable.*
(3) $\mathsf{Sep} \subseteq \mathsf{Prop} - \mathsf{LCan}$, $\mathsf{Sep} \subseteq \mathsf{Perf} - \mathsf{LCan}$.
(4) $\mathsf{Sep} - \mathsf{LCan} = \mathsf{Mor}$.

Remarks 2.9. (1) The property (c) which makes a pre-closed class into a closed class is somehow circumvented in the approach developed in [25], where, relative to a pre-closed class $\mathbb{F}$, the relevant class of "separated" morphisms is defined to be $(\mathbb{F} - \mathsf{Prop}) - \mathsf{Sep}$.
(2) What we call here a stable class is called a *topology* in [14].

Many examples of closed classes arise in the following way:

Definition 2.10. A *pre-closed pair* $(\mathbb{F}_0, \mathbb{F})$ on $\mathcal{C}$ consists of two classes of morphisms with $F_0 \subseteq \mathbb{F}$ and $\mathsf{Iso} \subseteq \mathbb{F}_0 \subseteq \mathsf{Mono}$ such that the following conditions hold:

(a) $\mathbb{F}$ is closed under compositions.
(b) $\mathbb{F}_0$ is pullback-stable and closed under composition.

Morphisms in $\mathbb{F}_0$ are called *closed immersions* and morphisms in $\mathbb{F}$ are called *closed morphisms*. A pre-closed pair is called a *closed pair* if

(c) $\mathbb{F} - \mathsf{Sep} \subseteq \mathbb{F}_0 - \mathsf{Sep}$.

Proposition 2.11. *Let $(\mathbb{F}_0, \mathbb{F})$ be a pre-closed pair on $\mathcal{C}$. We have:*

(1) $\mathbb{F}$ *is a pre-closed class on $\mathcal{C}$.*
(2) $\mathbb{F}_0 \subseteq \mathsf{Perf}$.

Let $(\mathbb{F}_0, \mathbb{F})$ be a closed pair on $\mathcal{C}$. We have:

(1) $\mathbb{F}$ *is a closed class on $\mathcal{C}$.*
(2) $\mathsf{Sep} \subseteq \mathbb{F}_0 - \mathsf{LCan}$.

Proof. (1) Since $\mathbb{F}_0 \subseteq \mathbb{F}$ and $\mathbb{F}_0$ is pullback-stable, we have $\mathbb{F}_0 \subseteq \mathbb{F} - \mathsf{Prop}$ and thus $\mathbb{F} - \mathsf{Sep} \subseteq \mathbb{F}_0 - \mathsf{Sep} \subseteq (\mathbb{F} - \mathsf{Prop}) - \mathsf{Sep}$. □

Example 2.12. In the category Top of topological spaces, take for $\mathbb{F}$ the closed morphisms and for $\mathbb{F}_0$ the closed embeddings. Then $(\mathbb{F}_0, \mathbb{F})$ is a closed pair on Top, and the resulting notions of separated and proper maps are the standard ones for topological spaces [2]. More generally, we will see in Section 3.2 that with respect to a proper factorization system $(\mathbb{E}, \mathbb{M})$, an $(\mathbb{E}, \mathbb{M})$-closed class $\mathbb{F}$ in the sense of [4] yields a closed pair $(\mathbb{F}_0 = \mathbb{F} \cap \mathbb{M}, \mathbb{F})$.

Example 2.13. In the category Sch of schemes, take for $\mathbb{F}$ the closed morphisms and for $\mathbb{F}_0$ the closed immersions (see Section 3.6). It is well known that $\mathbb{F} - \mathsf{Sep} \subseteq \mathbb{F}_0 - \mathsf{Sep}$ so $(\mathbb{F}_0, \mathbb{F})$ is a closed pair. In Sch, a morphism is called *separated* if it is $\mathbb{F}$-separated, it is called *universally closed* if it is $\mathbb{F}$-proper, and it is called *proper* if it is $\mathbb{F}$-perfect and of finite type. Let $\mathbb{P}$ be the class of morphisms of finite type. Then $\mathbb{P}$ is closed under composition and is pullback-stable. Furthermore, $\mathbb{F}_0 \subseteq \mathbb{P}$. Hence, $\mathsf{Sep} \subseteq \mathbb{P} - \mathsf{Sep}$ and Proposition 2.8(5) applies. Consequently, $\mathsf{Sep} \subseteq \mathbb{P} - \mathsf{LCan}$ and $\mathsf{Sep} \subseteq (\mathbb{P} \cap \mathsf{Perf}) - \mathsf{LCan}$, and we recover the fact that separated morphisms of schemes are left cancellable with respect to the proper maps of schemes. It is then readily seen that Propositions 2.8 and 2.11 remain valid with Perf replaced by $\mathbb{P} \cap \mathsf{Perf}$.

Example 2.14. In the category Sch, let $\mathbb{P}$ be the class of quasi-compact morphisms. Then $\mathsf{Iso} \subseteq \mathbb{P}$ and $\mathbb{P}$ is pullback-stable and closed under compositions. A morphism is called *quasi-separated* if it is in $\mathbb{P} - \mathsf{Sep}$. Thus Lemma 2.6 yields that quasi-separated morphisms are left cancellable with respect to quasi-compact morphisms.

2.4. Comparison functors

In this section, we investigate some elementary properties related to a finite limit preserving functor $\varphi : \mathcal{C} \longrightarrow \mathcal{C}'$ between finitely complete categories. Let $\mathbb{F}'$ be a class of morphisms in $\mathcal{C}'$, and consider the class $\varphi^{-1}(\mathbb{F}')$ on $\mathcal{C}$.

Proposition 2.15. (1) $\varphi^{-1}(\mathbb{F}' - \mathsf{Sep}) = \varphi^{-1}(\mathbb{F}') - \mathsf{Sep}$.
(2) $\varphi^{-1}(\mathbb{F}' - \mathsf{Prop}) \subseteq \varphi^{-1}(\mathbb{F}') - \mathsf{Prop}$.

Proof. (1) easily follows from the fact that for $f : X \longrightarrow Y$ in $\mathcal{C}$, the image under φ of the diagonal $\Delta_f : X \longrightarrow X \times_Y X$ in $\mathcal{C}$ is given by the diagonal $\varphi(\Delta_f) = \Delta_{\varphi(f)} : \varphi(X) \longrightarrow \varphi(X) \times_{\varphi(Y)} \varphi(X)$ in $\mathcal{C}'$. For (2), suppose $\varphi(f)$ is $\mathbb{F}'$-proper and consider a pullback

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

in $\mathcal{C}$. Since φ preserves the pullback, $\varphi(f')$ is a pullback of the $\mathbb{F}'$-proper morphism $\varphi(f)$, whence $\varphi(f') \in \mathbb{F}'$. □

Remark 2.16. Note that the converse inclusion in Proposition 2.15 is hard to control in general, since it requires one to consider arbitrary pullbacks of $\varphi(f)$ in $\mathcal{C}'$.

Corollary 2.17. *Let* $(\mathbb{F}_0, \mathbb{F})$ *be a closed pair on* $\mathcal{C}$ *and* $(\mathbb{F}'_0, \mathbb{F}')$ *a closed pair on* $\mathcal{C}'$. *Suppose:*

(1) *One of the following holds:*
  (a) $\varphi^{-1}(\mathbb{F}') \subseteq \mathbb{F}$.
  (b) $\varphi^{-1}(\mathbb{F}' - \mathsf{Prop}) \subseteq \mathbb{F} - \mathsf{Prop}$.
(2) *One of the following holds:*
  (a) $\mathbb{F}_0 \subseteq \varphi^{-1}(\mathbb{F}'_0)$
  (b) $\mathbb{F} \subseteq \varphi^{-1}(\mathbb{F}')$
  (c) $\mathbb{F} - \mathsf{Prop} \subseteq \varphi^{-1}(\mathbb{F}' - \mathsf{Prop})$

Then it follows that:

(1) $\varphi^{-1}(\mathbb{F}' - \mathsf{Sep}) = \mathbb{F} - \mathsf{Sep}$.
(2) $\varphi^{-1}(\mathbb{F}' - \mathsf{Prop}) \subseteq \mathbb{F} - \mathsf{Prop}$.

Example 2.18. Let FSch be the category of finite type schemes over $\mathbb{C}$, endowed with the same closed morphisms and closed immersions as Sch in Example 2.13, and let An be the category of complex analytic spaces [10]. Let $(-)^{an} : \mathsf{FSch} \longrightarrow \mathsf{An}$ be the analytification functor [24], and $|-| : \mathsf{An} \longrightarrow \mathsf{Top}$ the forgetful functor to the underlying topological space. We consider the composed functor

$$\varphi = |(-)^{an}| : \mathsf{FSch} \longrightarrow \mathsf{Top}.$$

One of the fundamental motivations for the adequate nature of the notions of properness and separatedness in algebraic geometry, is the fact that a morphism $f : X \longrightarrow Y$ of finite type schemes is separated (resp. proper) in the standard sense of Example 2.13 if and only if $|f^{an}|$ is separated (resp. perfect) in Top with the standard notions of Example 3.20. The entire dictionary between notions in FSch and An in [24] is based upon the subtle [24, Proposition 2.2] involving constructible sets. Although closed morphisms and universally closed morphisms are not explicitly considered in the dictionary, [24, Proposition 2.2] readily yields that closed morphisms are reflected by φ. Since by construction, closed immersions of schemes are mapped to closed analytic subspaces by analytification, and hence to closed embeddings under φ, conditions (1)(a) and (2)(a) in Corollary 2.17 are fulfilled, and thus it already follows that separated maps are preserved and reflected, and proper maps are reflected to universally closed maps. We should note however, that the direct proofs of these facts using [24, Proposition 2.2] are equally immediate. On the other hand, the fact that proper maps of schemes (with finite type included in the definition of properness) get mapped to perfect maps under φ requires an entirely different proof, based upon the familiar behavior of projective morphisms and Chow's Lemma.

3. Functional topology using factorizations

In [4], a general functional topology framework was set up against a "background" factorization system on the category. This approach has some advantages. First of all, if one has natural images of morphisms, one can formulate (and prove) intuitive statements like the fact that "the image of a closed subobject under a closed morphism is again a closed subobject". In fact, such an approach can be situated half-way between the basic setup of Section 2 and the richer setting of categories endowed with closure operators [7,5]. Secondly, factorization systems provide a rich source of functional topology examples. Indeed, if a category has a natural class of "closed subobjects", the higher intuitive statement can often be taken as the definition of a class of closed morphism.

However, if we want to apply these ideas for instance to the category of schemes, we are faced with a number of difficulties. This category does have a natural notion of closed subobjects, the closed immersions, but there is no natural class of general (i.e. not necessarily closed) subobjects which is part of a factorization system. Indeed, it is well known that for a morphism of schemes, the image of the underlying morphism of topological spaces can obviously be endowed with the subspace topology, but it cannot be endowed further with a scheme structure. In contrast, the closed immersions *are* the second class of a factorization system on schemes, the first class being given by the scheme theoretically dominant morphisms. If we compare this with the setting of a category with an "original" factorization system and a closure operator, which gives rise to a second "shadow" factorization system of dense morphisms and closed embeddings, then it is as if we find ourselves in a situation where only the shadow factorization system is present and everything else has vanished.

In this section, we extend the theory of [4] in such a way that examples like the category of schemes are naturally included. Our inspiration comes from the situation where we have two factorization systems as just described (see Section 3.4). If we think for instance about topological spaces, then we have the original factorization system of surjections followed by embeddings, and the shadow factorization system of dense morphisms followed by closed embeddings. A morphisms of topological spaces has a closed image precisely when the two factorizations coincide. To express this, we can take either of the two factorization systems as primary: we can say that in the original factorization, the embedding has to be a closed embedding, or equivalently we can say that in the shadow factorization, the dense morphism has to be a surjection. Generalizing these two possibilities at once, we define a *closed structure* (with respect to a background factorization system) using the input of *two* special classes of morphisms: a first class of morphism, called *surjections*, and a second class of morphisms, called *closed embeddings* (see Definition 3.12). With respect to a closed structure, a morphism is called *closed* if and only if for every closed embedding landing in the domain, the image factorization of the composition is given by a surjection followed by a closed embedding. We show that the closed embeddings and the closed morphisms together constitute a pre-closed pair in the sense of Section 2, and that some other familiar properties from [4] carry over to this setup. Using the forgetful functor from schemes to topological spaces, in Section 3.8 we show that the surjective morphisms of schemes and the closed immersions of schemes constitute a closed structure which yields the correct classes of proper and separated morphisms.

In order to obtain an understanding of the relation between the closed morphisms provided by this approach on the one hand, and morphisms with underlying closed morphism of topological spaces on the other hand, we are faced with the problem that the image factorization in Sch is not necessarily mapped to the (dense map, closed embedding) factorization in Top by the forgetful functor. For morphisms where this does hold true, our approach actually yields the correct notion of closedness. In Section 3.6, we identify a broad class of morphisms – encompassing for instance all quasi-compact morphisms – for which this applies. Our treatment makes use of the larger category LRS, and a relative "schematization" closure operator on the inclusion $\mathsf{Sch} \longrightarrow \mathsf{LRS}$, which we introduce on the way (Section 3.5).

3.1. Factorization systems

In this section, we recall some facts about factorization systems [8,1]. Let $\mathcal{C}$ be a finitely complete category.

Definition 3.1 (*[1]*). A *factorization system* $(\mathbb{E}, \mathbb{M})$ consists of two classes of morphisms, a first class $\mathbb{E}$ and a second class $\mathbb{M}$, such that:

(F1) $\mathbb{E}$ and $\mathbb{M}$ are closed under composition with isomorphisms.
(F2) Every $\mathbb{E}$-morphism is $\mathbb{M}$-orthogonal.
(F3) Every morphism f decomposes as $f = me$ with $m \in \mathbb{M}$, $e \in \mathbb{E}$.

Proposition 3.2 (*[1]*). *Let $(\mathbb{E}, \mathbb{M})$ be a factorization system. We have:*

(1) $\mathbb{E} = \mathbb{M} - \mathsf{Ortho}$.
(2) $\mathbb{E} \cap \mathbb{M} = \mathsf{Iso}$.
(3) $(\mathbb{M}^{op}, \mathbb{E}^{op})$ *is a factorization system on* $\mathcal{C}^{op}$.

Next we list a few properties of $\mathbb{M}$. By Proposition 3.2(3), the dual properties hold for $\mathbb{E}$.

Proposition 3.3 (*[1]*). *Suppose* $\mathbb{M}$ *is the second class of morphisms of a factorization system. Then we have* $\mathsf{Iso} \subseteq \mathbb{M}$, $\mathbb{M}$ *is closed under compositions and is pullback-stable,* $\mathbb{M} \subseteq \mathbb{M} - \mathsf{LCan}$, $\mathsf{Mono} \subseteq \mathbb{M} - \mathsf{LCan}$.

We recall the following useful fact:

Proposition 3.4 (*[1, Proposition 14.11]*). *Let $(\mathbb{E}, \mathbb{M})$ be a factorization system. The following are equivalent:*

(1) $\mathbb{E} \subseteq \mathsf{Epi}$.
(2) *All sections are contained in* $\mathbb{M}$.

Lemma 3.5. *Let* $\mathbb{M} \subseteq \mathsf{Mono}$ *with* $\mathbb{M}$ *pullback-stable. Then* $\mathbb{M} - \mathsf{Ortho} = \mathbb{M} - \mathsf{Dense}$.

Proof. See [4, Lemma 2.2].

Lemma 3.6. *Let* $\mathbb{M} \subseteq \mathsf{Mono}$ *be a class of morphisms. For a factorization* $f = mu$ *with* $m \in \mathbb{M}$*, consider the following properties:*

(a) *The* $\mathbb{M}$*-subobject* m *is minimal among the* $\mathbb{M}$*-subobjects* m' *for which there exists a factorization* $f = m'u'$.
(b) *The morphism* u *is* $\mathbb{M}$*-dense.*

We have:

(1) *If* $\mathbb{M}$ *is closed under composition, then (a) implies (b).*
(2) *If* $\mathbb{M} \subseteq \mathbb{M} - \mathsf{LCan}$*, then (b) implies (a).*

Proof. Suppose first (a). We show that u is dense. Write $u = m'u'$ with $m' \in \mathbb{M}$. Then $f = mm'u'$ with $mm' \in \mathbb{M}$ since $\mathbb{M}$ is closed under compositions. From the minimality of m we deduce that m' is an isomorphism as desired.

Suppose next (b). We show that m is minimal. Hence, consider $m' \in \mathbb{M}$ such that there are factorizations $f = m'u'$ and $m' = mn$. By $\mathbb{M}$-left cancellability of $\mathbb{M}$, $n \in \mathbb{M}$. Now $f = mnu'$ whence since m is mono, $u = nu'$. Now density of u yields that n is an isomorphism as desired. □

We obtain the following useful criteria for $\mathbb{M} \subseteq \mathsf{Mono}$ to be part of a factorization system:

Lemma 3.7. *Let* $\mathbb{M} \subseteq \mathsf{Mono}$ *and let* $\mathbb{E} = \mathbb{M} - \mathsf{Dense}$.

(1) *Suppose* $\mathbb{M}$ *is pullback-stable and is closed under composition with isomorphisms and* $(\mathbb{E}, \mathbb{M})$ *satisfies (F3). Then* $(\mathbb{E}, \mathbb{M})$ *is a factorization system.*
(2) *Suppose* $\mathsf{Iso} \subseteq \mathbb{M}$ *and that* $\mathbb{M}$ *is pullback-stable and closed under composition. Furthermore, suppose for every morphism* $f \in \mathcal{C}$ *there is a factorization* $f = mu$ *such that* $m \in \mathbb{M}$ *is minimal among the* $\mathbb{M}$*-subobjects* m' *for which there exists a factorization* $f = m'u'$*. Then* $(\mathbb{E}, \mathbb{M})$ *is a factorization system and the factorization* $f = mu$ *as above is the* $(\mathbb{E}, \mathbb{M})$*-factorization of* f.

3.2. $(\mathbb{E}, \mathbb{M})$-closed classes

In this section, we discuss $(\mathbb{E}, \mathbb{M})$-closed classes as introduced by Clementino, Giuli and Tholen [4]. Let $\mathcal{C}$ be a finitely complete category endowed with a factorization system $(\mathbb{E}, \mathbb{M})$ with $\mathbb{M} \subseteq \mathsf{Mono}$ and $\mathbb{E} \subseteq \mathsf{Epi}$.

Definition 3.8 (*[4]*). An $(\mathbb{E}, \mathbb{M})$-*closed class* $\mathbb{F}$ is a class of morphisms such that the following conditions hold:

(a) $\mathsf{Iso} \subseteq \mathbb{F}$ and $\mathbb{F}$ is closed under compositions.
(b) $\mathbb{F} \cap \mathbb{M}$ is pullback-stable.
(c) $\mathbb{E} \subseteq \mathbb{F} - \mathsf{RCan}$.

Proposition 3.9. *If the class* $\mathbb{F}$ *satisfies (a) and (b) in Definition 3.8, then we have that* $(\mathbb{F}_0 = \mathbb{F} \cap \mathbb{M}, \mathbb{F})$ *is a closed pair on* $\mathcal{C}$.

Proof. Conditions (a) and (b) in Definition 2.10 are fulfilled by definition. Since $\mathbb{E} \subseteq \mathsf{Epi}$, By Proposition 3.4, $\mathbb{M}$ contains the sections and hence $\mathbb{F} - \mathsf{Sep} \subseteq (\mathbb{F} \cap \mathbb{M}) - \mathsf{Sep}$. □

Remark 3.10. Condition (c) in Definition 3.8 mainly ensures the intuitive fact that "under a closed map, the image of a closed subobject is a closed subobject".

Example 3.11 (*[4, Exercise 2.1.4]*). Let $\mathsf{Iso} \subseteq \mathbb{F}_0 \subseteq \mathbb{M}$ with $\mathbb{F}_0$ pullback-stable and closed under composition. Define the morphism class $\mathbb{F}$ by $(f : X \longrightarrow Y) \in \mathbb{F}$ if and only if for every $m : X' \longrightarrow X$ in $\mathbb{F}_0$, in the $(\mathbb{E}, \mathbb{M})$-factorization $fm = \mu\epsilon$ with $\mu \in \mathbb{M}$ and $\epsilon \in \mathbb{E}$ we have $\mu \in \mathbb{F}_0$. Then $\mathbb{F}_0 = \mathbb{F} \cap \mathbb{M}$, $\mathbb{F}$ satisfies (a) and (b) in Definition 3.8, and $\mathbb{M}$ is $\mathbb{F}$-left cancellable. Furthermore, if $\mathbb{E}$ is stable under pullbacks along morphisms in $\mathbb{F}_0$, then $\mathbb{F}$ is an $(\mathbb{E}, \mathbb{M})$-closed class.

In the next section, rather than generalizing Definition 3.8, we will generalize Example 3.11 in order to be able to capture the situation in certain categories of ringed spaces.

3.3. $(\mathbb{E}, \mathbb{M})$-*closed structures*

Let $\mathcal{C}$ be a finitely complete category endowed with a factorization system $(\mathbb{E}, \mathbb{M})$ with $\mathbb{M} \subseteq \mathsf{Mono}$. From now on, we denote the $(\mathbb{E}, \mathbb{M})$-factorization of a morphism $f : X \longrightarrow Y$ by

$$X \xrightarrow[\epsilon(f)]{} f(X) \xrightarrow[\mu(f)]{} Y.$$

Definition 3.12. An $(\mathbb{E}, \mathbb{M})$-*pre-closed structure* $(\mathbb{P}, \mathbb{F}_0)$ (or simply *pre-closed structure* if $(\mathbb{E}, \mathbb{M})$ is understood) consists of two classes of morphisms such that:

(a) $\mathsf{Iso} \subseteq \mathbb{F}_0 \subseteq \mathbb{M}$ and $\mathbb{F}_0$ closed under compositions and pullback-stable.
(b) $\mathsf{Iso} \subseteq \mathbb{P}$ and $\mathbb{P}$ is closed under compositions.

Morphisms in $\mathbb{F}_0$ are called *closed embeddings* and morphisms in $\mathbb{P}$ are called *surjections*. With respect to a pre-closed structure, a morphism $f : X \longrightarrow Y$ is called *closed* if and only if for every $m : X' \longrightarrow X$ in $\mathbb{F}_0$, we have $\epsilon(fm) \in \mathbb{P}$ and $\mu(fm) \in \mathbb{F}_0$. The class of closed morphisms is denoted by $\mathbb{F}$. The pre-closed structure is called a *closed structure* if the following condition is satisfied:

(c) $\mathbb{F} - \mathsf{Sep} \subseteq \mathbb{M} - \mathsf{Sep}$.

Proposition 3.13. *Let $(\mathbb{P}, \mathbb{F}_0)$ be an $(\mathbb{E}, \mathbb{M})$-pre-closed structure. We have:*

(1) *$\mathbb{F}$ is closed under compositions, thus $(\mathbb{F}_0, \mathbb{F})$ is a pre-closed pair.*
(2) $\mathbb{M} \subseteq \mathbb{F} - \mathsf{LCan}$.
(3) $\mathbb{F}_0 = \mathbb{F} \cap \mathbb{M}$.
(4) *$\mathbb{F}$ is closed under pullback along morphisms in $\mathbb{F}_0$.*

Let $(\mathbb{P}, \mathbb{F}_0)$ be an $(\mathbb{E}, \mathbb{M})$-closed structure. We have:

(5) $\mathbb{F} - \mathsf{Sep} = \mathbb{F}_0 - \mathsf{Sep}$.
(6) *$(\mathbb{F}_0, \mathbb{F})$ is a closed pair.*

Proof. For a composition gf and $m \in \mathbb{M}$ we consider the $(\mathbb{E}, \mathbb{M})$-factorizations:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle m}\uparrow & & \uparrow{\scriptstyle \mu} & & \uparrow{\scriptstyle \mu'} \\ X' & \xrightarrow[\epsilon]{} & Y' & \xrightarrow[\epsilon']{} & Z' \end{array}$$

(1) Suppose $f, g \in \mathbb{F}$ and $m \in \mathbb{F}_0$. Then from $f \in \mathbb{F}$ we obtain $\epsilon \in \mathbb{P}$ and $\mu \in \mathbb{F}_0$ and from g $\in \mathbb{F}$ we obtain $\epsilon' \in \mathbb{P}$ and $\mu' \in \mathbb{F}_0$. Now $\epsilon'\epsilon$ and μ' constitute the $(\mathbb{E}, \mathbb{M})$-factorization of gfm and since $\mathbb{P}$ is closed under compositions, it follows that $\epsilon'\epsilon \in \mathbb{P}$ and $\mu' \in \mathbb{F}_0$ as desired.

(2) Suppose $g \in \mathbb{M}$, $gf \in \mathbb{F}$ and $m \in \mathbb{F}_0$. Now ϵ and $g\mu$ constitute the $(\mathbb{E}, \mathbb{M})$-factorization of gfm. Consequently, $\epsilon \in \mathbb{P}$ and $g\mu \in \mathbb{F}_0$. Since $g \in \mathbb{M} \subseteq \mathsf{Mono}$ and $\mathbb{F}_0$ pullback-stable and closed under compositions, by Lemma 2.6 g is $\mathbb{F}_0$-left cancellable whence also $\mu \in \mathbb{F}_0$.

(3) Suppose $f, m \in \mathbb{F}_0$. Then $fm \in \mathbb{F}_0 \subseteq \mathbb{M}$ hence $\mu(fm) = fm \in \mathbb{F}_0$ and $\epsilon(fm) = 1_{X'} \in \mathbb{P}$. It follows that $f \in \mathbb{F} \cap \mathbb{M}$. Conversely, if $f \in \mathbb{F} \cap \mathbb{M}$ we consider 1_X as $\mathbb{F}_0$-subobject. It follows that $f = \mu(f 1_X) \in \mathbb{F}_0$.

(4) Consider the pullback square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle m'}\uparrow & & \uparrow{\scriptstyle m} \\ P & \xrightarrow[f']{} & Y' \end{array}$$

with $f \in \mathbb{F}$, $m \in \mathbb{F}_0$. Then $mf' = fm' \in \mathbb{F}$ whence by (1), $f' \in \mathbb{F}$.

(5) Immediate from (c) and (3).

(6) Immediate by (a), (1) and (4). □

Remarks 3.14. (1) Suppose in Definition 3.12 we have $\mathbb{E} \subseteq \mathsf{Epi}$. Then by Proposition 3.4, $\mathsf{Mor} \subseteq \mathbb{M} - \mathsf{Sep}$ so condition (c) becomes automatic.

(2) Suppose we take $\mathbb{P} = \mathbb{E}$. Then a morphism $f : X \longrightarrow Y$ is closed if and only if for $m : X' \longrightarrow X$ in $\mathbb{F}_0$ we have $\mu(fm) \in \mathbb{F}_0$. Thus, this is precisely the situation of Example 3.11.

(3) Suppose we take $\mathbb{F}_0 = \mathbb{M}$. Then a morphisms $f : X \longrightarrow Y$ is closed if and only if for $m : X' \longrightarrow X$ in $\mathbb{F}_0$ we have $\epsilon(fm) \in \mathbb{P}$.

We finally note that the following does not make use of the assumption $\mathbb{E} \subseteq \mathsf{Epi}$ in Example 3.11:

Proposition 3.15. *Let $(\mathbb{E}, \mathbb{F}_0)$ be an $(\mathbb{E}, \mathbb{M})$-pre-closed structure. If $\mathbb{E}$ is closed under pullbacks along $\mathbb{F}_0$, then $\mathbb{E}$ is $\mathbb{F}$-right cancellable.*

3.4. Two factorization systems

Let $\iota : \mathcal{C} \longrightarrow \mathcal{D}$ be the inclusion of a full subcategory, and suppose both categories are finitely complete. Suppose $\mathcal{D}$ is endowed with a factorization system $(\mathbb{E}, \mathbb{M})$ and $\mathcal{C}$ with a factorization system $(\mathbb{D}, \mathbb{F}_0)$ such that $\mathbb{F}_0 \subseteq \mathbb{M} \subseteq \mathsf{Mono}_{\mathcal{D}}$. Consequently, $\mathbb{F}_0 \subseteq \mathsf{Mono}_{\mathcal{C}}$, $\mathbb{E} = \mathbb{M} - \mathsf{Dense}$ in $\mathcal{D}$, $\mathbb{D} = \mathbb{F}_0 - \mathsf{Dense}$ in $\mathcal{C}$ and $\mathbb{E} \cap \mathcal{C} \subseteq \mathbb{D}$. For a morphism $f : X \longrightarrow Y$ in $\mathcal{D}$ we denote the $(\mathbb{E}, \mathbb{M})$-factorization as in Section 3.3 and for f in $\mathcal{C}$ we denote the $(\mathbb{D}, \mathbb{F}_0)$-factorization by

$$X \xrightarrow[\delta(f)]{} \overline{f(X)} \xrightarrow[\varphi(f)]{} Y.$$

Lemma 3.16. *For a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$, the following are equivalent:*

(1) *The $(\mathbb{E}, \mathbb{M})$-factorization and the $(\mathbb{D}, \mathbb{F}_0)$-factorization of f are isomorphic.*
(2) $\mu(f) \in \mathbb{F}_0$.
(3) $\delta(f) \in \mathbb{E}$.

Put $\mathbb{E}' = \mathbb{E} \cap \mathcal{C}$.

Proposition 3.17. *We have that $(\mathbb{E}', \mathbb{F}_0)$ is a $(\mathbb{D}, \mathbb{F}_0)$-pre-closed structure, and the following are equivalent for $f : X \longrightarrow Y$ in $\mathcal{C}$:*

(1) *f is closed.*
(2) *For every $m : X' \longrightarrow X$ in $\mathbb{F}_0$, the $(\mathbb{E}, \mathbb{M})$-factorization and the $(\mathbb{D}, \mathbb{F}_0)$-factorization of fm are isomorphic.*

In case $\iota = 1_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$, $(\mathbb{E}, \mathbb{F}_0)$ is also an $(\mathbb{E}, \mathbb{M})$-pre-closed structure and f is closed with respect to $(\mathbb{E}, \mathbb{M})$ if and only if it is closed with respect to $(\mathbb{D}, \mathbb{F}_0)$. Further, $(\mathbb{E}, \mathbb{F}_0)$ is an $(\mathbb{E}, \mathbb{M})$-closed structure if and only if it is a $(\mathbb{D}, \mathbb{F}_0)$-closed structure.

Let $\mathbb{F}$ denote the class of closed morphisms as described in Proposition 3.17

Lemma 3.18. $\mathbb{F} \cap \mathbb{D} \subseteq \mathbb{E}$.

Proof. Consider $f \in \mathbb{F} \cap \mathbb{D}$. Since $f \in \mathbb{F}$, we have $\mu(f) = \varphi(f)$ and $\epsilon(f) = \delta(f)$. But since $f \in \mathbb{D}$, $\varphi(f)$ is an isomorphism and hence $f \cong \epsilon(f) \in \mathbb{E}$. □

Example 3.19. The situation (with $\iota = 1_{\mathcal{C}}$) applies if $(\mathbb{E}, \mathbb{M})$ is a factorization system on $\mathcal{C}$ and $\mathbb{F}_0$ is the class of closed subobjects with respect to an idempotent, weakly hereditary closure operator on $\mathcal{C}$ in the sense of [7]. In Section 3.5, we will present a notion of relative closure operator which yields examples with $\iota \neq 1_{\mathcal{C}}$.

Example 3.20. On Top with its Kuratowski closure, we make the following choices:

- $\mathbb{M}$ is the class of embeddings.
- $\mathbb{E}$ is the class of surjective morphisms.
- $\mathbb{F}_0$ is the class of closed embeddings.
- $\mathbb{D}$ is the class of dense morphisms.

The $(\mathbb{E}, \mathbb{M})$-factorization of $f : X \longrightarrow Y$ arises from endowing the set theoretic image $f(X)$ with the subspace topology from Y, and the $(\mathbb{D}, \mathbb{F}_0)$-factorization of f arises from endowing the closure $\overline{f(X)}$ with the subspace topology from Y.

Then $(\mathbb{E}, \mathbb{F}_0)$ is an $(\mathbb{E}, \mathbb{M})$-closed structure by Proposition 3.17 and Remark 3.14 (1), and consequently $(\mathbb{E}, \mathbb{F}_0)$ is a $(\mathbb{D}, \mathbb{F}_0)$-closed structure with the same class $\mathbb{F}$ of closed maps. Clearly, these are precisely the usual closed maps i.e. morphisms mapping closed subsets to closed subsets. Lemma 3.18 now becomes the familiar fact that a morphism which is closed and dense is surjective.

3.5. Relative closure operators

In this section we collect some facts concerning closure operators relative to an inclusion functor in the sense of [3]. Let $\iota : \mathcal{C} \longrightarrow \mathcal{D}$ be the inclusion of a full subcategory where $\mathcal{D}$ is endowed with a factorization system $(\mathbb{E}, \mathbb{M})$ with $\mathbb{M} \subseteq \mathsf{Mono}(\mathcal{D})$. We suppose both categories are finitely complete. We suppress ι in all notations. For $X \in \mathcal{C}$, we denote by $\mathsf{Sub}_{\mathcal{C}}(X)$ the $\mathbb{M}$-subobjects in $\mathcal{C}$ and by $\mathsf{Sub}_{\mathcal{D}}(X)$ the $\mathbb{M}$-subobjects in $\mathcal{D}$. For a morphism $f : X \longrightarrow Y$ in $\mathcal{D}$ and $m \in \mathsf{Sub}_{\mathcal{D}}(X)$, the *image* of m under f is by definition the $\mathbb{M}$-part of the $(\mathbb{E}, \mathbb{M})$-factorization of the composition fm, and is denoted by $f(m)$. The *image* of f is by definition the image of 1_X under f, and is denoted $f(X) = f(1_X)$.

Definition 3.21. Let $c = (c_X)_{X\in\mathcal{C}}$ consist of a family of maps

$$c_X : \mathrm{Sub}_{\mathcal{D}}(X) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(X).$$

(1) For $X \in \mathcal{C}$, a subobject $m \in \mathrm{Sub}_{\mathcal{C}}(X)$ is *c-closed* if $c_X(m) = m$.
(2) For $X \in \mathcal{C}$, a subobject $m \in \mathrm{Sub}_{\mathcal{D}}(X)$ is *c-dense* if $c_X(m) = 1_X$.
(3) A morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is *c-closed* if for every c-closed $m \in \mathrm{Sub}_{\mathcal{C}}(X)$, the image $f(m)$ is c-closed as well.
(4) A morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is *c-dense* if the image $f(X) = f(1_X)$ is c-dense, i.e. $c_Y(f(1_X)) = 1_Y$.

Consider the following conditions:

(1) (*extension*) For $X \in \mathcal{C}$ and $m \in \mathrm{Sub}_{\mathcal{D}}(X)$, we have $m \leq c_X(m)$. In this case we denote the canonical $\mathbb{M}$-morphism from the domain of m to the domain of $c_X(m)$ by $b_X(m)$, i.e we have $m = c_X(m)b_X(m)$.
(2) (*monotonicity*) For $X \in \mathcal{C}$ and $m \leq m'$ in $\mathrm{Sub}_{\mathcal{D}}(X)$, we have $c_X(m) \leq c_X(m')$ in $\mathrm{Sub}_{\mathcal{C}}(X)$.
(3) (*continuity*) For $f : X \longrightarrow Y$ in $\mathcal{C}$ and $m \in \mathrm{Sub}_{\mathcal{D}}(X)$, we have $f(c_X(m)) \leq c_Y(f(m))$.
(4) (*idempotency*) For $X \in \mathcal{C}$ and $m \in \mathrm{Sub}_{\mathcal{D}}(X)$, we have $c_X(c_X(m)) = c_X(m)$.
(5) (*weak heredity*) For $X \in \mathcal{C}$ and $m \in \mathrm{Sub}_{\mathcal{D}}(X)$, $b_X(m)$ is a c-dense subobject of the domain of $c_X(m)$.

If c satisfies (1) and (2), it is called a *local closure operator on* ι. If c further satisfies (3), it is called a *closure operator on* ι. A (local) closure operator is called *idempotent* if it satisfies (4) and *weakly hereditary* if it satisfies (5).

Remark 3.22. The definition of c-closed maps given in (3) is most useful for idempotent closure operators. For a non idempotent closure operator, it is more important to understand how the image preserves closures, rather than closed subobjects.

Like for usual closure operators, we have the following familiar characterizations of continuity:

Proposition 3.23. *Let c be a local closure operator on $\iota : \mathcal{C} \longrightarrow \mathcal{D}$. Consider the following conditions:*

(3) *c satisfies (3), i.e. c is a closure operator.*
(3′) *For $f : X \longrightarrow Y$ in $\mathcal{C}$ and $n \in \mathrm{Sub}_{\mathcal{D}}(Y)$, we have $c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n))$.*
(3″) *For every c-closed $n \in \mathrm{Sub}_{\mathcal{C}}(Y)$, we have that $f^{-1}(n) \in \mathrm{Sub}_{\mathcal{D}}(X)$ is c-closed as well.*

Conditions (3) and (3′) are equivalent. If c is idempotent, these conditions are further equivalent to (3″).

Inspired by [7], one can use certain closure operators to construct factorization systems on $\mathcal{C}$.

Proposition 3.24. *Let $\iota : \mathcal{C} \longrightarrow \mathcal{D}$ and $(\mathbb{E}, \mathbb{M})$ be as above. Let c be an idempotent, weakly hereditary closure operator on ι. Let $\mathbb{M}_c$ consist of the c-closed $\mathbb{M}$-subobjects of $\mathcal{C}$-objects and let $\mathbb{E}_c$ consist of the c-dense $\mathcal{C}$-morphisms.*

Then $(\mathbb{E}_c, \mathbb{M}_c)$ is a factorization system on $\mathcal{C}$, and for a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ with $(\mathbb{E}, \mathbb{M})$-factorization given by $f = \mu(f)\epsilon(f)$, the $(\mathbb{E}_c, \mathbb{M}_c)$-factorization is given by $\mu_c(f) = c_Y(\mu(f))$ and $\epsilon_c(f) = b_Y(\mu(f))\epsilon(f)$.

Proof. In the proposed $(\mathbb{E}_c, \mathbb{M}_c)$-factorization, the map $\mu_c(f) = c_Y(\mu(f))$ is c-closed by idempotency of c and $\epsilon_c(f) = b_Y(\mu(f))\epsilon(f)$ is c-dense since c is weakly hereditary. Using all the properties of a closure operator, it is further shown that the c-dense morphisms are $\mathbb{M}_c$-orthogonal. □

Clearly, the results of Section 3.4 apply and we obtain:

Proposition 3.25. *Let $\iota : \mathcal{C} \longrightarrow \mathcal{D}$, $(\mathbb{E}, \mathbb{M})$ on $\mathcal{D}$ and $(\mathbb{E}_c, \mathbb{M}_c)$ on $\mathcal{C}$ be as above and put $\mathbb{E}' = \mathbb{E} \cap \mathcal{C}$. Then $(\mathbb{E}', \mathbb{M}_c)$ is an $(\mathbb{E}_c, \mathbb{M}_c)$-pre-closed structure on $\mathcal{C}$ and a $\mathcal{C}$-morphism is closed with respect to this structure if and only if it is c-closed.*

At the other extreme from applications with $\iota = 1_{\mathcal{C}}$, we have the following interesting situation. Let $\iota : \mathcal{C} \longrightarrow \mathcal{D}$ and $(\mathbb{E}, \mathbb{M})$ on $\mathcal{D}$ be as above. Suppose that for every $X \in \mathcal{C}$ and $m \in \mathrm{Sub}_{\mathcal{D}}(X)$, there exists a smallest $m' \in \mathrm{Sub}_{\mathcal{C}}(X)$ with $m \leq m'$. Put $c_X(m)$ equal to this smallest subobject.

Proposition 3.26. *We have that $c = (c_X)_{X\in\mathcal{C}}$ is an idempotent, weakly hereditary local closure operator on ι. If for every $f : X \longrightarrow Y$ in $\mathcal{C}$ and $m \in \mathrm{Sub}_{\mathcal{C}}(Y)$, the $\mathcal{D}$-pullback $f^{-1}(m)$ is in $\mathcal{C}$, c is an idempotent, weakly hereditary closure operator.*

We have $\mathbb{M}_c = \mathbb{M} \cap \mathcal{C}$, and an arbitrary $\mathcal{C}$-morphism f is c-closed if and only if for every $\mathbb{M}$-subobject in $\mathcal{C}$, the image $f(m)$ (based upon the $(\mathbb{E}, \mathbb{M})$-factorization of fm) lies in $\mathcal{C}$.

If, for a given $\iota : \mathcal{C} \longrightarrow \mathcal{D}$ the local closure operator described in Proposition 3.26 exists, we denote it by c_ι. We end this section by illustrating the special role of c_ι. Let c be an arbitrary local closure operator on ι. By restricting the domains of the maps

$$c_X : \mathrm{Sub}_{\mathcal{D}}(X) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(X)$$

to $\mathrm{Sub}_{\mathcal{C}}(X)$, we obviously obtain a local closure operator $c_{\mathcal{C}}$ on $\mathcal{C}$, which inherits all the properties from c. We now have:

Proposition 3.27. *Suppose c is idempotent. Then we have*

$$c = c_{\mathcal{C}} \circ c_\iota,$$

i.e. for every $m \in \mathrm{Sub}_{\mathcal{D}}(X)$ with $X \in \mathcal{C}$, we have $c_X(m) = (c_{\mathcal{C}})_X(c_\iota)_X(m)$.

Proof. We omit the subscript X from the notation. Since $m \leq c_\iota(m)$, we have $c(m) \leq c(c_\iota(m)) = c_{\mathcal{C}}(c_\iota(m))$. Conversely, by definition of c_ι, we have $c_\iota(m) \leq c(m)$. Thus, it follows that $c_{\mathcal{C}}(c_\iota(m)) \leq c_{\mathcal{C}}(c(m)) = c(c(m)) = c(m)$. □

3.6. Application to schemes

Let LRS be the category of locally ringed spaces and Sch the category of schemes. In this section, we present an application of Section 3.5 to the inclusion $\iota : \mathsf{Sch} \longrightarrow \mathsf{LRS}$. To start, we will endow the category LRS with a factorization system.

Remark 3.28. The essential property of locally ringed spaces that we use is the fact that $\mathrm{supp}(\mathcal{O}_X) = \{x \in X \mid (\mathcal{O}_X)_x \neq 0\} = X$ or, equivalently, $\mathcal{O}_X(V) \neq 0$ for $V \neq \varnothing$. Everything goes through with LRS replaced by the category of ringed spaces satisfying this property.

Let $(X, \mathcal{O}_X)$ be a locally ringed space. There is a corresponding Grothendieck category $\mathsf{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules. Let $\mathrm{Sub}(\mathcal{O}_X)$ be the lattice of subobjects of $\mathcal{O}_X$ in $\mathsf{Mod}(\mathcal{O}_X)$, i.e. the lattice of ideal sheaves $\mathcal{I} \subseteq \mathcal{O}_X$. Let LRS/X be the category of morphisms $f : Y \longrightarrow X$ in LRS.

We consider the following map:

$$\mathcal{I} : \mathrm{Ob}(\mathsf{LRS}/X) \longrightarrow \mathrm{Sub}(\mathcal{O}_X) : (f : Y \longrightarrow X) \longmapsto \mathcal{I}(f) = \mathrm{Ker}(f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y).$$

In the category LRS, we recall the following special type of subobjects.

Definition 3.29. A morphism $f : Y \longrightarrow X$ of locally ringed spaces is a *closed immersion* if the following conditions hold:

(1) $f : Y \longrightarrow X$ is a homeomorphism of Y onto a closed subspace $f(Y) \subseteq X$.
(2) $f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is an epimorphism of sheaves.

A closed immersion of locally ringed spaces is a monomorphism in LRS. If $f : Y \longrightarrow X$ is a closed immersion, we call Y a closed subobject (or closed subspace) of X. Let $\mathrm{Sub}_{cl}(X)$ denote the poset of (isomorphism classes of) closed subobjects of X. We obtain a restriction

$$\mathcal{I} : \mathrm{Sub}_{cl}(X) \longrightarrow \mathrm{Sub}(\mathcal{O}_X) : (i : Z \longrightarrow X) \longmapsto \mathcal{I}(Z) = \mathcal{I}(i).$$

Example 3.30. Let X be a locally ringed space and let $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal. We describe an associated subspace $Z(\mathcal{I})$ with $\mathcal{I}(Z(\mathcal{I})) = \mathcal{I}$.

The sheaf of rings $\mathcal{O}_X/\mathcal{I}$ is finitely generated and hence

$$Z = \mathrm{supp}(\mathcal{O}_X/\mathcal{I}) = \{x \in X \mid (\mathcal{O}_X)_x/\mathcal{I}_x \neq 0\}$$

is a closed subset of X. For the inclusion $i : Z \longrightarrow X$ we put $\mathcal{O}_Z = i^{-1}(\mathcal{O}_X/\mathcal{I})$. It follows that $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$ whence we can put $i^\sharp : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z$ equal to the canonical quotient map. Clearly, this is a morphism of locally ringed spaces and we have made Z into a closed subspace of X with associated ideal given by $\mathcal{I}$.

Proposition 3.31. *Let $f : Y \longrightarrow X$ be a morphism of locally ringed spaces and let $i : Z \longrightarrow X$ be a closed immersion. The following are equivalent:*

(1) *There is a unique morphism $g : Y \longrightarrow Z$ with $f = ig$.*
(2) $\mathcal{I}(i) \subseteq \mathcal{I}(f)$.

Proof. Since i is a monomorphism, unicity of g will be automatic.

Suppose first the existence of g. We thus obtain $f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ as a composition $f^\sharp \cong i_*(g^\sharp)i^\sharp$. Thus, $\mathcal{I}(i) = \mathrm{Ker}(i^\sharp) \subseteq \mathrm{Ker}(f^\sharp) = \mathcal{I}(f)$.

Suppose next that $\mathcal{I}(i) \subseteq \mathcal{I}(f)$. We may assume that $Z \subseteq X$ is a closed subspace. We have $f(Y) \subseteq \mathrm{supp}(f_*\mathcal{O}_Y)$ and $Z = \mathrm{supp}(i_*\mathcal{O}_Z)$. Since $i^\sharp : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z$ is surjective, the inclusion between ideals yields a factorization $f^\sharp = \varphi i^\sharp$ for some $\varphi : i_*\mathcal{O}_Z \longrightarrow f_*\mathcal{O}_Y$. Hence, if $(i_*\mathcal{O}_Z)_x = 0$ we have $f^\sharp_x = 0$ and hence $(f_*\mathcal{O}_Y)_x = 0$. This already shows that $f(Y) \subseteq Z$. Denote by $g : X \longrightarrow X$ the natural factorization such that $f = ig$. We have $f_*\mathcal{O}_Y \cong i_*g_*\mathcal{O}_Y$. On the level of sheaves of rings, we define $g^\sharp = i^{-1}(\varphi)$. Now $f = ig$ as morphisms of ringed spaces, as desired. □

Recall that a *duality* between posets is an order reversing isomorphism.

Proposition 3.32. *The map $\mathcal{I}$ defines a duality*

$$\mathcal{I} : \mathrm{Sub}_{cl}(X) \longrightarrow \mathrm{Sub}(\mathcal{O}_X) : (i : Z \longrightarrow X) \longmapsto \mathcal{I}(\mathcal{Z}).$$

Its inverse is given by the construction in Example 3.30.

Proof. The construction in Example 3.30 shows $\mathcal{I}$ to be surjective. Furthermore, by Proposition 3.31 we have $Z_1 \leq Z_2$ if and only if $\mathcal{I}(Z_2) \subseteq \mathcal{I}(Z_1)$. □

For a morphism $f : Y \longrightarrow X$ in LRS, we call the subspace $Z(\mathcal{I}(f)) \subseteq X$ the *locally ringed image of* f.

Let $\mathbb{M}$ be the class of closed immersions and $\mathbb{E}$ the class of $\mathbb{M}$-dense morphisms.

Proposition 3.33. $(\mathbb{E}, \mathbb{M})$ *constitutes a factorization system on* LRS. *The image of a morphism with respect to this factorization system is the locally ringed image.*

Proof. This follows from Lemma 3.7(2) by Proposition 3.31. □

Next we take a closer look at the class $\mathbb{E}$.

Definition 3.34. A morphism $f : Y \longrightarrow X$ is called *ringed dominant* if $f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is a monomorphism.

Proposition 3.35. *A morphism of locally ringed spaces is $\mathbb{M}$-dense if and only if it is ringed dominant.*

Proof. By definition f is $\mathbb{M}$-dense if and only if the only closed immersion through which f factors is an isomorphism. By Proposition 3.31, this corresponds to the fact that the only ideal inside $\mathcal{I}(f)$ is the zero ideal. But this means that $\mathcal{I}(f) = 0$, or, equivalently, that $f^\sharp$ is a monomorphism. □

Proposition 3.36. *If $f : Y \longrightarrow X$ is $\mathbb{M}$-dense, the underlying morphism of topological spaces is dense, i.e. $\overline{f(Y)} = X$.*

Proof. If f is not topologically dense, there is an open subset $\varnothing \neq U \subseteq X$ with $U \cap f(Y) = \varnothing$. Hence, $f_*\mathcal{O}_Y(U) = \mathcal{O}_Y(f^{-1}(U)) = \mathcal{O}_Y(\varnothing) = 0$. But since $\mathcal{O}_X(U) \neq 0$, $\mathcal{O}_X(U) \longrightarrow f_*\mathcal{O}_Y(U)$ cannot be a monomorphism. □

Next we turn our attention to the full subcategory $\mathsf{Sch} \subseteq \mathsf{LRS}$ of schemes, i.e. locally ringed spaces that are locally isomorphic to spectra of commutative rings. A *closed immersion* of schemes is a morphism of schemes which is a closed immersion in LRS. If $f : Y \longrightarrow X$ is a closed immersion of schemes, we call Y a closed subscheme of X. We denote by $\mathrm{Sub}_{sch}(X)$ the poset of (isomorphism classes of) closed subschemes of X.

We recall the following:

Proposition 3.37 (*[9, Proposition 10.30]*)**.** *For a quasi-compact morphism $f : Y \longrightarrow X$ of schemes, the sheaf $\mathcal{I}(f) = \mathrm{Ker}(f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y)$ is quasi-coherent.*

Lemma 3.38. *Let X be a scheme and $Z \in \mathrm{Sub}_{cl}(X)$ a locally ringed subspace. The following are equivalent:*

(1) *Z is a scheme.*
(2) *$\mathcal{I}(Z)$ is a quasi-coherent sheaf.*

Proof. By Proposition 3.37, (1) implies (2). The converse implication easily reduces to the case where $X = \mathrm{Spec}(A)$ is affine. In this case $\mathsf{Qch}(X) \cong \mathsf{Mod}(A)$ and in the construction of Example 3.30, an ideal $I \subseteq A$ gives rise to the corresponding scheme $Z(I) = \mathrm{Spec}(A/I)$. □

Let $\mathrm{Sub}_{qch}(\mathcal{O}_X)$ be the lattice of subobjects of $\mathcal{O}_X$ in the category $\mathsf{Qch}(X)$ of quasi-coherent sheaves on X.

Proposition 3.39. *The map $\mathcal{I}$ defines a duality*

$$\mathcal{I} : \mathrm{Sub}_{sch}(X) \longrightarrow \mathrm{Sub}_{qch}(\mathcal{O}_X) : Z \longrightarrow \mathcal{I}(Z).$$

Now we are in the right position to define a closure operator on the inclusion $\iota : \mathsf{Sch} \longrightarrow \mathsf{LRS}$.

Proposition 3.40. *There is an idempotent, weakly hereditary closure operator* sch *on ι with, for $X \in \mathsf{Sch}$ and $Z \in \mathrm{Sub}_{cl}(X)$,*

$$\mathrm{sch}(Z) = \inf\{Z' \in \mathrm{Sub}_{sch}(X) \mid Z \subseteq Z'\} \in \mathrm{Sub}_{sch}(X).$$

Proof. For the closure operator to be well defined, we have to make sure that $\mathrm{Sub}_{sch}(X)$ is closed under infima in $\mathrm{Sub}_{cl}(X)$. Dually, it suffices that $\mathrm{Sub}_{qch}(\mathcal{O}_X)$ is closed under suprema in $\mathrm{Sub}(\mathcal{O}_X)$. But in both Grothendieck categories $\mathsf{Qch}(X)$ and $\mathsf{Mod}(X)$, direct sums and images coincide, and a supremum $\sup_i F_i$ of subobjects is obtained as image of the canonical morphism $\oplus_i F_i \longrightarrow \mathcal{O}_X$. Thus, the proposition is an application of Proposition 3.26 by Lemma 3.41. □

Lemma 3.41. *Let $f : Y \longrightarrow X$ be a morphism in LRS and consider $Z \in \mathrm{Sub}_{cl}(X)$ with ideal $\mathcal{I} = \mathcal{I}(Z)$. The pullback $f^{-1}Z$ in LRS is the closed subspace corresponding to the ideal $f^*(\mathcal{I}) \subseteq f^*\mathcal{O}_X = \mathcal{O}_Y$ for the inverse image functor $f^* : \mathsf{Mod}(X) \longrightarrow \mathsf{Mod}(Y)$. In particular, if f is a morphism of schemes and $Z \in \mathrm{Sub}_{sch}(X)$, we have $f^{-1}Z \in \mathrm{Sub}_{sch}(Y)$.*

Proof. The second claim follows from the fact that f^* preserves quasi-coherence of sheaves. □

In Sch, let $\mathbb{M}_{sch} = \mathbb{M} \cap \mathsf{Sch}$ be the class of closed immersions between schemes and $\mathbb{E}_{sch}$ the class of $\mathbb{M}_{sch}$-dense (or equivalently sch-dense) morphisms. Let $\mathbb{E}' = \mathbb{E} \cap \mathsf{Sch}$ be the class of ringed dominant morphisms between schemes. For a morphism $f : Y \longrightarrow X$ in Sch with locally ringed image Z, $\mathrm{sch}(Z)$ is called the *scheme theoretic image of f* (see [11,9]).

Proposition 3.42. *$(\mathbb{E}_{sch}, \mathbb{M}_{sch})$ constitutes a factorization system on Sch and $(\mathbb{E}', \mathbb{M}_{sch})$ is an $(\mathbb{E}_{sch}, \mathbb{M}_{sch})$-closed structure on Sch. The following are equivalent for a morphism $f : Y \longrightarrow X$ of schemes:*

(1) *f is closed with respect to $(\mathbb{E}', \mathbb{M}_{sch})$.*
(2) *f is* sch*-closed.*
(3) *For every $m \in \mathrm{Sub}_{sch}(Y)$, the locally ringed image of fm is a scheme (and hence coincides with the scheme theoretic image).*

Proof. This follows from Propositions 3.25 and 3.26. □

Example 3.43. A quasi-compact morphism $f : Y \longrightarrow X$ is sch-closed. Indeed, for every $m \in \mathrm{Sub}_{sch}(Y)$, the composition fm remains quasi-compact and thus by Proposition 3.37, the locally ringed image is a scheme, hence coincides with the scheme theoretic image. In particular, in this case Y is topologically dense in the scheme theoretic image. This property fails in general, as [9, Exercise 10.18] shows.

Remarks 3.44. (1) In a similar fashion, the inclusion of the subcategory $\mathsf{An} \subseteq \mathsf{LRS}$ of analytic spaces can be endowed with a closure operator. A detailed analysis of functional topology in this category is work in progress and will appear in a subsequent work.

(2) The fact that $\mathbb{M}_{sch}$ is part of a factorization system on Sch was also used in [20], without explicit reference to the larger category of locally ringed spaces.

3.7. *Forgetful functors*

In this section we consider a functor $|\cdot| : \mathcal{C} \longrightarrow \mathcal{A}$ between finitely complete categories. Note that we do *not* require this functor to preserve finite limits.

We suppose that $\mathcal{A}$ is endowed with a factorization system $(\mathcal{D}, \mathcal{F}_0)$ and a $(\mathcal{D}, \mathcal{F}_0)$-pre-closed structure $(\mathcal{P}, \mathcal{F}_0)$ for which $\mathcal{P} \subseteq \mathcal{D}$. Note that this applies to the situation described in Section 3.4. Let $\mathcal{F}$ be the class of closed morphisms.

We suppose further that $\mathcal{C}$ is endowed with a factorization system $(\mathbb{D}, \mathbb{F}_0)$ and that the following holds:

$(*)$ $|\mathbb{F}_0| \subseteq \mathcal{F}_0$ and every $a : A \longrightarrow |X|$ in $\mathcal{F}_0$ is isomorphic to $|m| : |X'| \longrightarrow |Y|$ with $m : X' \longrightarrow X$ in $\mathbb{F}_0$.

We denote $(\mathbb{D}, \mathbb{F}_0)$-factorizations by $f = \varphi(f)\delta(f)$.

Now define:

(1) $\mathbb{P} = \{f \in \mathcal{C} \mid |f| \in \mathcal{P}\}$.
(2) $\mathbb{H} = \{f \in \mathcal{C} \mid |f| \in \mathcal{F}\}$.
(3) $\mathbb{D}_0 = \{f \in \mathcal{C} \mid |f| \in \mathcal{D}\}$.

Both $(\mathbb{D}_0, \mathbb{F}_0)$ and $(\mathbb{P}, \mathbb{F}_0)$ are readily seen to be $(\mathbb{D}, \mathbb{F}_0)$-pre-closed structures. Denote the class of $(\mathbb{D}_0, \mathbb{F}_0)$-closed morphisms by $\mathbb{K}$ and the class of $(\mathbb{P}, \mathbb{F}_0)$-closed structures by $\mathbb{F}$.

Proposition 3.45. *We have:*

(1) $\mathbb{F} \subseteq \mathbb{H}$.
(2) $\mathbb{F} = \mathbb{H} \cap \mathbb{K}$.
(3) *If* $\mathbb{H} - \mathsf{Prop} \subseteq \mathbb{K}$, *then* $\mathbb{H} - \mathsf{Prop} = \mathbb{F} - \mathsf{Prop}$.
(4) *If* $\mathbb{H} - \mathsf{Sep} \subseteq \mathbb{F}_0 - \mathsf{Sep}$, *then* $\mathbb{H} - \mathsf{Sep} = \mathbb{F} - \mathsf{Sep} = \mathbb{F}_0 - \mathsf{Sep}$ *and* $(\mathbb{P}, \mathbb{F}_0)$ *is a* $(\mathbb{D}, \mathbb{F}_0)$*-closed structure.*

Proof. (1) Let $f : X \longrightarrow Y$ in $\mathbb{F}$ and consider $|f| : |X| \longrightarrow |Y|$. By the assumption $(*)$, an arbitrary $\mathcal{F}_0$-subobject of $|X|$ is given by $|m| : |X'| \longrightarrow |X|$ for some $m : X' \longrightarrow X$ in $\mathbb{F}_0$. Consider the $(\mathbb{D}, \mathbb{F}_0)$-factorization

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle m}\uparrow & & \uparrow{\scriptstyle \varphi} \\ X' & \xrightarrow[\delta]{} & Y'. \end{array} \qquad (1)$$

By definition, we have $\delta \in \mathbb{P}$ and $\varphi \in \mathbb{F}_0$. Thus $|\delta| \in \mathcal{P} \subseteq \mathcal{D}$ and $|\varphi| \in \mathcal{F}_0$ is the $(\mathcal{D}, \mathcal{F}_0)$-factorization of $|f||m|$ and since $|\delta| \in \mathcal{P}$ it follows that $|f| \in \mathcal{F}$ as desired.

(2) Let $f : X \longrightarrow Y$ in $\mathbb{H} \cap \mathbb{K}$ and consider $m : X' \longrightarrow X$ in $\mathbb{F}_0$. By assumption, in the factorization (1) we have $|\delta| \in \mathcal{D}$ and we have $|\varphi| \in \mathcal{F}_0$. Hence, $|\delta|$ and $|\varphi|$ constitute the $(\mathcal{D}, \mathcal{F}_0)$ factorization of $|f||m|$ and since $|f|$ is closed and $|m| \in \mathcal{F}_0$, it follows that $|\delta| \in \mathcal{P}$ whence $\delta \in \mathbb{P}$ as desired.

(3) Since $(\mathbb{P}, \mathbb{F}_0)$ is a $(\mathbb{D}, \mathbb{F}_0)$-pre-closed structure and by (1), we have $\mathbb{F}_0 \subseteq \mathbb{F} \subseteq \mathbb{H}$ and hence $\mathbb{F}_0 \subseteq \mathbb{F} - \mathsf{Prop} \subseteq \mathbb{H} - \mathsf{Prop}$. Since $\mathbb{H} \in \mathsf{CC}$, we thus have for every pullback of f and for every $\mathbb{F}_0$-subobject of the domain that the composition is in $\mathbb{H} - \mathsf{Prop}$. Thus, it suffices to show that $f \in \mathbb{H} - \mathsf{Prop}$ implies $f \in \mathbb{F}$. Also, for every $\mathbb{F}_0$-subobject m we have that fm satisfies the condition in (2), hence it follows that $f \in \mathbb{F}$.

(4) Immediate from the inclusions $\mathbb{F}_0 \subseteq \mathbb{F} \subseteq \mathbb{H}$. $\square$

3.8. Application to schemes

Let $\mathcal{C} = \mathsf{Sch}$ be the category of schemes and consider the forgetful functor $|\cdot| : \mathsf{Sch} \longrightarrow \mathsf{Top}$. On $\mathcal{A} = \mathsf{Top}$, we take:

- $\mathcal{F}_0$ the class of closed embeddings.
- $\mathcal{D}$ the class of dense morphisms.
- $\mathcal{P}$ the class of surjective morphisms.

By Example 3.20, $(\mathcal{P}, \mathcal{F}_0)$ is a $(\mathcal{D}, \mathcal{F}_0)$-closed structure and $\mathcal{F}$ is the class of standard closed morphisms. On Sch, we take $\mathbb{F}_0$ to be the class of closed immersions. Then $\mathbb{F}_0 \subseteq \mathsf{Mono}$ and according to Section 3.6, $(\mathbb{D} = \mathbb{F}_0 - \mathsf{Dense}, \mathbb{F}_0)$ is a factorization system on Sch. With the notations of Section 3.7, $\mathbb{H}$ consists of the closed morphisms of schemes (note that this class was denoted by $\mathbb{F}$ in Example 2.13), $\mathbb{P}$ consists of the surjective morphisms of schemes, $\mathbb{D}_0$ consists of the morphisms of schemes that are dense on the topological level. Further, $\mathbb{F}$ is the class of $(\mathbb{P}, \mathbb{F}_0)$-closed morphisms and $\mathbb{K}$ is the class of $(\mathbb{D}_0, \mathbb{F}_0)$-closed morphisms. Finally, let $\mathbb{E}'$ be the class of ringed dominant morphisms and $\mathbb{L}$ the class of sch-closed morphisms from Section 3.6.

Proposition 3.46. *We have:*

(1) $\mathbb{F} \subseteq \mathbb{H}$.
(2) $\mathbb{F} = \mathbb{H} \cap \mathbb{K}$.
(3) $\mathbb{H} - \mathsf{Prop} = \mathbb{F} - \mathsf{Prop}$.
(4) $\mathbb{H} - \mathsf{Sep} = \mathbb{F} - \mathsf{Sep} = \mathbb{F}_0 - \mathsf{Sep}$.
(5) $(\mathbb{P}, \mathbb{F}_0)$ *is a* $(\mathbb{D}, \mathbb{F}_0)$*-closed structure.*
(6) $\mathbb{L} \subseteq \mathbb{K}$.

Proof. (6) By Proposition 3.36, $\mathbb{E}' \subseteq \mathbb{D}_0$.

For the other claims, it suffices to check the conditions in Proposition 3.45.

(3) Morphisms in $\mathbb{H}$-Prop are quasi-compact, so the result follows from Example 3.43 and (6).

(4) A $\mathbb{H}$-separated morphism is well-known to be $\mathbb{F}_0$-separated. □

Remark 3.47. As recalled in Example 2.13, in Sch, a morphism is called *separated* if it is $\mathbb{H}$-separated. It is called *universally closed* if it is $\mathbb{H}$-proper, and it is called *proper* it is $\mathbb{H}$-perfect and of finite type. Thus, the $(\mathbb{D}, \mathbb{F}_0)$-closed structure $(\mathbb{P}, \mathbb{F}_0)$ yields the correct class of separated morphisms and the correct class of universally closed morphisms, called proper in our terminology. When we restrict our attention to quasi-compact morphisms (or to the sch-closed or $(\mathbb{D}_0, \mathbb{F}_0)$-closed morphisms), it also yields the correct class of closed morphisms.

3.9. Comparison functors

Let $\varphi : \mathcal{C} \longrightarrow \mathcal{C}'$ be a finite limit preserving functor between finitely complete categories, and suppose $\mathcal{C}$ is endowed with an $(\mathbb{E}, \mathbb{M})$-closed structure $(\mathbb{P}, \mathbb{F}_0)$ with corresponding class $\mathbb{F}$ of closed morphisms and $\mathcal{C}'$ is endowed with an $(\mathbb{E}', \mathbb{M}')$-closed structure $(\mathbb{P}', \mathbb{F}'_0)$with corresponding class $\mathbb{F}'$ of closed morphisms. We then have the following way of obtaining that φ reflects closed morphisms:

Proposition 3.48. *In either of the following cases, we have* $\varphi^{-1}(\mathbb{F}') \subseteq \mathbb{F}$*:*

(1) $\mathbb{X} \subseteq \varphi^{-1}(\mathbb{X}')$ *for* $\mathbb{X} = \mathbb{F}_0, \mathbb{E}, \mathbb{M}$ *and* $\varphi^{-1}(\mathbb{X}') \subseteq \mathbb{X}$ *for* $\mathbb{X} = \mathbb{F}_0, \mathbb{P}$.
(2) $\mathbb{P} = \mathbb{E}$ *and* $\mathbb{X} \subseteq \varphi^{-1}(\mathbb{X}')$ *for* $\mathbb{X} = \mathbb{F}_0, \mathbb{E}, \mathbb{M}$ *and* $\varphi^{-1}(\mathbb{F}'_0) \subseteq \mathbb{F}_0$.
(3) $\mathbb{F}_0 = \mathbb{M}$ *and* $\mathbb{X} \subseteq \varphi^{-1}(\mathbb{X}')$ *for* $\mathbb{X} = \mathbb{E}, \mathbb{M}$ *and* $\varphi^{-1}(\mathbb{P}') \subseteq \mathbb{P}$.

Example 3.49. Consider the functor $\varphi : \mathsf{FSch} \longrightarrow \mathsf{Top}$ of Example 2.18. We take for $\mathbb{F}_0 = \mathbb{M}$ the class of closed immersions, for $\mathbb{E}$ the class of ringed dominant morphisms, and for $\mathbb{P}$ the class of surjections. We take for $\mathbb{F}'_0 = \mathbb{M}'$ the class of closed embeddings, for $\mathbb{E}'$ the class of dense morphisms and for $\mathbb{P}'$ the class of surjections. By construction, closed immersions are mapped to closed embeddings, and by the dictionary in [24] based upon [24, Proposition 2.2], ringed dominant morphisms are mapped to dense morphisms and surjectivity is reflected by φ. Thus, we are in case (3) in Proposition 3.48 whence closed morphisms are reflected. We should note however, that the fact that closed morphisms are reflected follows directly from [24, Proposition 2.2] as well.

4. Presheaf categories

Although the extended functional topology theory developed in Section 3 encompasses the examples of schemes, we have lost some of the simplicity by the introduction of an additional input datum of so called surjective morphisms. Thus, we still wonder whether it is possible to recover the correct classes of proper and separated morphisms in these categories using *only* the original theory from [4]. In this section, we investigate this possibility by making use of presheaf categories. The main idea is the following: if $\mathcal{C}$ is the category we are interested in, and we have natural classes of proper and separated morphisms in $\mathcal{C}$, then we try to install the original functional topology setup from [4] on the category $\hat{\mathcal{C}}$ of presheaves of sets on $\mathcal{C}$. The category $\hat{\mathcal{C}}$ is well-suited for this purpose. Indeed, it comes equipped with the (Epi, Mono)-factorization system, and the class Epi is pullback stable. Thus, by Example 3.11 from [4], any pullback stable class $\mathbb{F}_0$ of monomorphisms which is closed under compositions is the class of closed embeddings for an (Epi, Mono)-closed class $\mathbb{F}$ in the sense of [4], and $\mathbb{F}$ consists precisely of the morphisms for which the image of a closed subobject is again a closed subobject.

Since we start with proposed classes of proper and separated morphisms on $\mathcal{C}$, our aim is to define a class $\mathbb{F}_0$ on $\hat{\mathcal{C}}$ such that a morphism in $\mathcal{C}$ is proper (resp. separated) if and only if it is proper (resp. separated) with respect to $\mathbb{F}$ when considered as a morphism in $\hat{\mathcal{C}}$. For $\mathbb{F}_0$, we propose to take precisely the images of proper-representable $\hat{\mathcal{C}}$-morphisms (see Section 4.2). After analyzing the general situation in Section 4.3, we prove in Section 4.4 that taking for $\mathcal{C}$ the category of schemes, this approach works. We also identify the $\mathbb{F}_0$-morphisms that belong to $\mathcal{C}$ as being precisely the proper monomorphisms. Thus, this class contains the ordinary closed immersions, and when we restrict our attention to morphisms between schemes of finite type, the two classes coincide. However, even in this case, and even for a closed morphism, the presheaf image will in general not be a scheme, and should not be confused with the scheme theoretic image which we used in Section 3.

4.1. $(\mathbb{E}, \mathbb{M})$-*closed structures from stable classes*

Let $\mathcal{C}$ be a finitely complete category endowed with a factorization system $(\mathbb{E}, \mathbb{M})$ with $\mathbb{M} \subseteq$ Mono and $\mathbb{E}$ pullback-stable. Let $\mathbb{H}$ be a stable class of morphisms in $\mathcal{C}$ (i.e. Iso $\subseteq \mathbb{H}$ and $\mathbb{H}$ closed under compositions and pullback-stable). We define the class

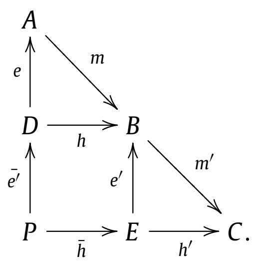
$$\mathbb{F}_0 = \{\mu(h) \mid h \in \mathbb{H}\}.$$

Proposition 4.1. $(\mathbb{E}, \mathbb{F}_0)$ *is an* $(\mathbb{E}, \mathbb{M})$-*pre-closed structure.*

Proof. Clearly, Iso $\subseteq \mathbb{F}_0 \subseteq \mathbb{M}$ and obviously Iso $\subseteq \mathbb{E}$ and $\mathbb{E}$ is closed under compositions.

Let us show first that $\mathbb{F}_0$ is pullback-stable. For $m \in \mathbb{F}_0$, we have an $h = me$ with $h \in \mathbb{H}$ and $e \in \mathbb{E}$. Taking a pullback of this yields $\bar{h} = \bar{m}\bar{e}$ with $\bar{h} \in \mathbb{H}$ and $\bar{e} \in \mathbb{E}$ by the assumptions. Thus, $\bar{m} \in \mathbb{F}_0$ as desired.

Let us now show that $\mathbb{F}_0$ is closed under compositions. For $m : A \longrightarrow B$ and $m' : B \longrightarrow C$ in $\mathbb{F}_0$ we write $h = me$ and $h' = m'e'$ with $h, h' \in \mathbb{H}$ and $e, e' \in \mathbb{E}$. Taking the pullback of f and e', we obtain the diagram

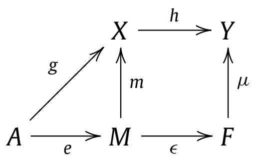


So we see that $h'\bar{h} = m'me\bar{e'}$ with $h'\bar{h} \in \mathbb{H}$. □

Let $\mathbb{F}$ be the class of closed morphisms associated to $(\mathbb{E}, \mathbb{F}_0)$.

Proposition 4.2. $\mathbb{H} \subseteq \mathbb{F} -$ Prop.

Proof. Let $h : X \longrightarrow Y$ in $\mathbb{H}$. Since $\mathbb{H}$ is pullback-stable, it suffices to show that $h \in \mathbb{F}$. So consider $m : M \longrightarrow X$ in $\mathbb{F}_0$. We obtain a diagram



with g and hence hg in $\mathbb{H}$, e and ϵ and hence ϵe in $\mathbb{E}$, and μ in $\mathbb{M}$. Hence, $\mu \in \mathbb{F}_0$ as desired. □

Remark 4.3. The inclusion $\mathbb{H} \subseteq \mathbb{F}$ expresses the intuitive fact that for a morphism in $\mathbb{H}$, the image of an $\mathbb{F}_0$-subobject is again an $\mathbb{F}_0$-subobject. This is comparable to the situation for $(\mathbb{E}, \mathbb{M})$-closed classes, see Remark 3.10.

4.2. *Representable morphisms*

In this section, starting from a closed class on a category, we define an associated stable class on an enlargement of the category.

Let $\mathcal{C} \subseteq \hat{\mathcal{C}}$ be a fully faithful left exact inclusion between finitely complete categories. Let $\mathbb{H}$ be a closed class of morphisms in $\mathcal{C}$. A morphism $f : X \longrightarrow Y$ in $\hat{\mathcal{C}}$ is called $\mathbb{H}$-*representable* if for every pullback

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow \\ P & \xrightarrow[f']{} & C \end{array}$$

with $C \in \mathcal{C}$, it follows that $P \in \mathcal{C}$ and $f' \in \mathbb{H}$. We denote the class of $\mathbb{H}$-representable morphisms by $\mathbb{H} - \mathsf{Rep}$. A morphism is called *representable* if it is $\mathsf{Mor}(\mathcal{C})$-representable and we denote $\mathsf{Rep} = \mathsf{Mor}(\mathcal{C}) - \mathsf{Rep}$.

Lemma 4.4. (1) $\mathsf{Iso} \subseteq \mathbb{H} - \mathsf{Rep}$.
(2) $\mathbb{H} - \mathsf{Rep}$ *is pullback-stable and closed under compositions.*
(3) $\mathbb{H} - \mathsf{Rep} \cap \mathsf{Mor}(\mathcal{C}) = \mathbb{H} - \mathsf{Prop}$.

Proof. Clear. □

4.3. Presheaf categories

Let $\mathcal{C}$ be a finitely complete category and consider the Yoneda embedding $\mathcal{C} \subseteq \hat{\mathcal{C}} = \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$. The category $\hat{\mathcal{C}}$ is endowed with the factorization system $(\mathsf{Epi}, \mathsf{Mono})$ and Epi is pullback stable. The image of a natural transformation between presheaves is computed pointwise. Let $\mathbb{H}$ be a closed class on $\mathcal{C}$ and consider the class $\mathbb{H} - \mathsf{Rep}$ on $\hat{\mathcal{C}}$. We define the following class of morphisms in $\hat{\mathcal{C}}$:

$$\mathbb{F}_0 = \{\mu(h) \mid h \in \mathbb{H} - \mathsf{Rep}\}.$$

Proposition 4.5. $(\mathsf{Epi}, \mathbb{F}_0)$ *is an* $(\mathsf{Epi}, \mathsf{Mono})$*-closed structure on* $\hat{\mathcal{C}}$. *Let* $\mathbb{F}$ *denote the associated class of closed morphisms.*

(1) $(\mathbb{H} - \mathsf{Prop})_{\mathcal{C}} \subseteq \mathbb{H} - \mathsf{Rep} \subseteq (\mathbb{F} - \mathsf{Prop})_{\hat{\mathcal{C}}}$.
(2) $(\mathbb{H} - \mathsf{Prop})_{\mathcal{C}} \cap \mathsf{Mono}_{\mathcal{C}} \subseteq \mathbb{H} - \mathsf{Rep} \cap \mathsf{Mono} \subseteq \mathbb{F}_0$.

Proof. According to Proposition 4.1, $(\mathsf{Epi}, \mathbb{F}_0)$ is an $(\mathsf{Epi}, \mathsf{Mono})$-pre-closed structure and according to Proposition 4.2, $\mathbb{H} - \mathsf{Rep} \subseteq (\mathbb{F} - \mathsf{Prop})_{\hat{\mathcal{C}}}$. The other inclusions in (1) and (2) easily follow. Finally, it is a closed structure by Remark 3.14(1). □

Remark 4.6. Note that the $(\mathsf{Epi}, \mathsf{Mono})$-closed structure $(\mathsf{Epi}, \mathbb{F}_0)$ on $\hat{\mathcal{C}}$ fits entirely into Example 3.11. In particular, $\mathbb{F}$ is an $(\mathsf{Epi}, \mathsf{Mono})$-closed class in the sense of [4], see Definition 3.8.

Under an additional hypothesis, we can characterize the representable $\mathbb{F}_0$-morphisms as follows:

Proposition 4.7. *Suppose* $\mathcal{C}$*-retractions are* $\mathbb{H}$*-right cancellable.*

(1) $\mathbb{H} - \mathsf{Rep} \cap \mathsf{Mono} = \mathbb{F}_0 \cap \mathsf{Rep}$.
(2) $(\mathbb{H} - \mathsf{Prop})_{\mathcal{C}} \cap \mathsf{Mono}_{\mathcal{C}} = \mathbb{F}_0 \cap \mathsf{Mor}(\mathcal{C})$.

Proof. (2) follows from (1) by intersecting with the $\mathcal{C}$-morphisms. For (1), the inclusion left in right follows from Proposition 4.5(2). Conversely, let $m : F \longrightarrow G$ be a representable morphism in $\mathbb{F}_0$. By taking the pullback along a $\mathcal{C}$-object, we may suppose that m is a $\mathcal{C}$-morphism in $\mathbb{F}_0$. Thus, there exists $h \in (\mathbb{H} - \mathsf{Prop})_{\mathcal{C}}$ with $h = me$ and e an epimorphism. Then by Lemma 2.6, e is representable and $e : H \longrightarrow F$ is a presheaf epimorphism between $\mathcal{C}$-objects, i.e. a $\mathcal{C}$-retraction. By the assumption, it follows that $m \in (\mathbb{H} - \mathsf{Prop})_{\mathcal{C}}$ as desired. □

Next we take a closer look at separated morphisms.

Proposition 4.8. *We have*

(1) $((\mathbb{H} - \mathsf{Prop})_{\mathcal{C}} - \mathsf{Sep})_{\mathcal{C}} \subseteq (\mathbb{F} - \mathsf{Sep})_{\hat{\mathcal{C}}} \cap \mathsf{Mor}(\mathcal{C}) = (\mathbb{F}_0 - \mathsf{Sep})_{\hat{\mathcal{C}}} \cap \mathsf{Mor}(\mathcal{C})$.

If $\mathcal{C}$*-retractions are* $\mathbb{H}$*-right cancellable, then*

(2) $((\mathbb{H} - \mathsf{Prop})_{\mathcal{C}} - \mathsf{Sep})_{\mathcal{C}} = (\mathbb{F} - \mathsf{Sep})_{\hat{\mathcal{C}}} \cap \mathsf{Mor}(\mathcal{C}) = (\mathbb{F}_0 - \mathsf{Sep})_{\hat{\mathcal{C}}} \cap \mathsf{Mor}(\mathcal{C})$.

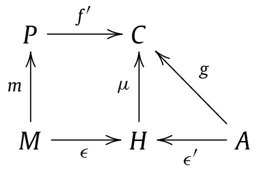
If $\mathbb{H}$ *is a stable class in* $\mathcal{C}$, *we can replace* $(\mathbb{H} - \mathsf{Prop})_{\mathcal{C}}$ *by* $\mathbb{H}$ *in the above.*

4.4. Application to schemes

Put $\mathcal{C} = \mathsf{Sch}$ in the previous section and take $\mathbb{H}$ to be the usual class of closed morphisms, i.e. morphisms with an underlying closed map of topological spaces. Let $\hat{\mathcal{C}}$, $\mathbb{F}_0$ and $\mathbb{F}$ be as above. Our aim is to prove that the $(\mathsf{Epi}, \mathsf{Mono})$-closed structure $(\mathsf{Epi}, \mathbb{F}_0)$ on $\hat{\mathcal{C}}$ yields the classes of $\mathbb{H}$-proper and $\mathbb{H}$-separated morphisms (after restricting to $\mathcal{C}$-morphisms). We start with properness.

Proposition 4.9. (1) $\mathbb{H} - \mathsf{Rep} = (\mathbb{F} - \mathsf{Prop})_{\hat{\mathcal{C}}} \cap \mathsf{Rep}$.
(2) $(\mathbb{H} - \mathsf{Prop})_{\mathcal{C}} = (\mathbb{F} - \mathsf{Prop})_{\hat{\mathcal{C}}} \cap \mathsf{Mor}(\mathcal{C})$.

Proof. (2) immediately follows from (1) by restricting to morphisms in $\mathcal{C}$. For (1), the inclusion from left to right follows from Proposition 4.2 (taking $\mathbb{H}$ to be $\mathbb{H} - \mathsf{Rep}$). Consider $f : F \longrightarrow G$ in $(\mathbb{F} - \mathsf{Prop})_{\hat{\mathcal{C}}} \cap \mathsf{Rep}$. Let $C \longrightarrow G$ be a morphism with $C \in \mathcal{C}$. We are to show that the pullback $f' : P \longrightarrow C$ is in $\mathbb{H}$. By assumption, $P \in \mathcal{C}$ and $f' \in \mathbb{F}$. For a closed subset $X \subseteq C$, let $m : M \longrightarrow C$ be a closed subscheme with $|M| = X$. By Proposition 4.5(2), $m \in \mathbb{F}_0$ and so by definition of $\mathbb{F}$ and $\mathbb{F}_0$, we obtain the following diagram



with $\mu \in \mathbb{F}_0 \cap \mathbb{M}$, $\epsilon \in \mathbb{E}$, $g \in \mathbb{H} - \mathsf{Rep}$ and $\epsilon' \in \mathbb{E}$. Since $C \in \mathcal{C}$, we necessarily have $g \in \mathbb{H}$. We are to show that the set theoretic image of the $\mathcal{C}$-morphism $f'm$ is a closed subset of C. But the presheaf images of $f'm$ and g coincide, whence by Lemma 4.10, so do the set theoretic images and we are ready. □

Lemma 4.10. *Consider two morphism $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y$ of schemes and suppose the presheaf images $\mu = \mu(f) = \mu(f') : F \longrightarrow Y$ coincide. Then the set theoretic images $f(X) \subseteq Y$ and $f'(X') \subseteq Y$ coincide as well.*

Proof. Clearly the situation is symmetric. Consider $\epsilon = \epsilon(f) : X \longrightarrow F$ and $\epsilon' = \epsilon(f') : X' \longrightarrow F$. Take a point $x \in X$ with residue field k. This point is represented by a morphism $\xi : \mathsf{Spec}(k) \longrightarrow X$ and its image $f(x) \in Y$ is determined by the composition $f\xi : \mathsf{Spec}(k) \longrightarrow X \longrightarrow F \longrightarrow Y$. Now since $\epsilon' : X' \longrightarrow F$ is an epimorphism, there is a morphism $\xi' : \mathsf{Spec}(k) \longrightarrow X'$ with $\mu\epsilon\xi = \mu\epsilon'\xi'$ and hence a point $x' \in X'$ with $f'(x') = y$. □

We have a precise description of the representable $\mathbb{F}_0$-morphisms, and we obtain the correct class of separated morphisms after restricting to $\mathcal{C}$-morphisms:

Proposition 4.11. (1) $\mathbb{F}_0 \cap \mathsf{Rep} = \mathbb{H} - \mathsf{Rep} \cap \mathsf{Mono}$.
(2) $\mathbb{F}_0 \cap \mathsf{Mor}(\mathcal{C}) = (\mathbb{H} - \mathsf{Prop})_{\mathcal{C}} \cap \mathsf{Mono}_{\mathcal{C}}$.
(3) $(\mathbb{F}_0 - \mathsf{Sep})_{\hat{\mathcal{C}}} \cap \mathsf{Mor}(\mathcal{C}) = (\mathbb{F} - \mathsf{Sep})_{\hat{\mathcal{C}}} \cap \mathsf{Mor}(\mathcal{C}) = (\mathbb{H} - \mathsf{Sep})_{\mathcal{C}}$.

Proof. Since retractions of schemes are surjective on the level of underlying spaces, they are $\mathbb{H}$-right cancellable and the condition in Propositions 4.7 and 4.8(2) is fulfilled. □

Remark 4.12. For an $\mathbb{H}$-proper morphism $f : X \longrightarrow Y$ of schemes, the presheaf image $\mu : F \longrightarrow Y$ belongs to $\mathbb{F}_0$. However, it should not be confused with the scheme theoretic image $m : Z \longrightarrow Y$ which also belongs to $\mathbb{F}_0$, where Z is a scheme rather than a presheaf, but where the corresponding $e : X \longrightarrow Z$ fails to be a presheaf epimorphism in general (the only scheme morphisms that are presheaf epimorphisms are the retractions).

Remark 4.13. Since the usual closed immersions of schemes are always proper monomorphisms, they are always contained in $\mathbb{F}_0$. For morphisms between schemes of finite type, the converse holds and the closed immersions are precisely the $\mathbb{F}_0$-morphisms between such schemes.

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