Available online at www.sciencedirect.com
SCIENCE DIRECT®
JOURNAL OF PURE AND APPLIED ALGEBRA
ELSEVIER
Journal of Pure and Applied Algebra 190 (2004) 197–211
www.elsevier.com/locate/jpaa

A generalization of the Gabriel–Popescu theorem

W. Lowen[1]

Department of Mathematics, Vrije Universiteit Brussel, Pleinlaan 2, Brussel 1050, Belgium

Received 31 January 2003; received in revised form 10 November 2003
Communicated by C.A. Weibel

Abstract

In this paper we give necessary and sufficient conditions for an additive functor $u:\mathfrak{u}\to\mathscr{C}$, from a small pre-additive category $\mathfrak{u}$ to a Grothendieck category $\mathscr{C}$, to realize $\mathscr{C}$ as a localization of the category of presheaves on $\mathfrak{u}$. This is a generalization of the Gabriel–Popescu theorem, which considers the case where u is fully faithful.

MSC: 18E15; 18E35; 18F10; 18F20

1. Introduction

A fully faithful functor between arbitrary categories is called a *localization* provided that it has a left adjoint commuting with finite limits.

Consider a small pre-additive category $\mathfrak{u}$ and the additive functor category of (additive) presheaves $\mathbf{Pr}(\mathfrak{u})=\mathbf{Add}(\mathfrak{u}^{\mathrm{op}},\mathbf{Ab})$. The Gabriel–Popescu theorem states that every Grothendieck category (i.e. a cocomplete abelian category with a generator and exact filtered colimits) $\mathscr{C}$ is a localization of a category $\mathbf{Pr}(\mathfrak{u})$.

Theorem 1.1 (Gabriel and Popescu [5], Takeuchi [10], Mitchell [8]). *Consider a Grothendieck category $\mathscr{C}$ and a full generating subcategory $\mathfrak{u}$. Then $\mathscr{C}\to\mathbf{Pr}(\mathfrak{u}):C\mapsto\mathscr{C}(-,C)$ is a localization.*[2]

[1] The author is an aspirant at the FWO.
E-mail address: wlowen@vub.ac.be (W. Lowen).
[2] This is in fact a slight generalization of the original Gabriel–Popescu theorem to families of generators. That such a generalization is possible is well-known.

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2003.11.016

However for a given Grothendieck category $\mathscr{C}$, the Gabriel–Popescu theorem does not allow us to construct *all* its realizations as localizations of categories of the form $\mathbf{Pr}(\mathfrak{u})$. The reason is that in general the natural functor from $\mathfrak{u}$ to a localization of $\mathbf{Pr}(\mathfrak{u})$ will be neither full nor faithful. Our aim in this paper will be to characterize those additive functors

$$u : \mathfrak{u} \to \mathscr{C}$$

from a small pre-additive category $\mathfrak{u}$ to a Grothendieck category $\mathscr{C}$ which do give rise to a localization.

To state this characterization it will be convenient to call a collection of maps $U_i \to U$ in $\mathfrak{u}$ *epimorphic* (with respect to u) if the induced map $\coprod_i u(U_i) \to u(U)$ is an epimorphism in $\mathscr{C}$.

(G) We say that u satisfies (G) if the objects $u(U)$ for U in $\mathfrak{u}$ form a generating family for $\mathscr{C}$.

(F) We say that u satisfies (F) if for every map $c : u(U) \to u(V)$ in $\mathscr{C}$ there exists an epimorphic collection $f_i : U_i \to U$ such that $cu(f_i)$ is in the image of u for all i.

(FF) We say that u satisfies (FF) if for every map $f : U \to V$ in the kernel of u there exists an epimorphic collection $f_i : U_i \to U$ such that $f f_i = 0$ for all i.

The following theorem is the main result of this paper.

Theorem 1.2. *Consider an additive functor $u : \mathfrak{u} \to \mathscr{C}$ from a small pre-additive category $\mathfrak{u}$ to a Grothendieck category $\mathscr{C}$. The following are equivalent*:

(1) *The functor $\mathscr{C} \to \mathbf{Pr}(\mathfrak{u}) : C \mapsto \mathscr{C}(u(-), C)$ is a localization.*
(2) *u satisfies the conditions* (G), (F) *and* (FF).

The nature of the conditions (G) and (F) and (FF) makes it natural to use sheaf-theoretic language to prove Theorem 1.2. This language will probably be most familiar in a non-additive context: Consider a small category $\mathfrak{u}$ and the functor category of presheaves $\mathbf{Pr}(\mathfrak{u}) = \mathbf{Fun}(\mathfrak{u}^{\mathrm{op}}, \mathbf{Set})$. It is well known that all localizations of $\mathbf{Pr}(\mathfrak{u})$ are obtained, up to equivalence, as categories of sheaves $\mathbf{Sh}(\mathfrak{u}, \mathscr{T})$ for Grothendieck topologies $\mathscr{T}$ on $\mathfrak{u}$ (roughly speaking a Grothendieck topology on $\mathfrak{u}$ is given by specifying a set of "coverings" for every object in $\mathfrak{u}$, see Section 2 for a precise definition in the additive context). A category equivalent to such a category of sheaves is called a Grothendieck topos and a theorem of Giraud gives an intrinsic characterization of Grothendieck topoi [1,2]. This sheaf-theoretic framework generalizes to certain enriched categories [3]. In this paper we work in an additive context (i.e. categories are enriched in $\mathbf{Ab}$).

Let $\mathfrak{u}$ be a small pre-additive category as above. It is well known that all localizations of $\mathbf{Pr}(\mathfrak{u}) = \mathbf{Add}(\mathfrak{u}^{\mathrm{op}}, \mathbf{Ab})$ are Grothendieck categories and can be obtained (up to equivalence) as categories of sheaves $\mathbf{Sh}(\mathfrak{u}, \mathscr{T})$ for (additive analogues) of Grothendieck topologies $\mathscr{T}$ on $\mathfrak{u}$ [3].

It follows that Grothendieck categories are nothing but the additive counterpart of Grothendieck topoi, and the Gabriel–Popescu theorem is simply an additive version of Giraud's theorem. So below we adapt the proof of Giraud's theorem to an additive context, taking at the same time into account the weakened hypotheses (F) and (FF).

We also give "relative" versions of conditions (G), (F) and (FF) that allow us to generalize (in the additive setting) the "Lemme de comparaison" [1] which gives sufficient conditions for a functor between small categories to yield an equivalence on the level of sheaf categories.

Finally, we illustrate conditions (G), (F) and (FF) in the example of sheaves of modules over a ringed space.

The main result in this paper is used in the forthcoming [7] which gives a general treatment of the deformation theory of abelian categories. We have decided to publish it separately since it may be of independent interest. In the first version of this paper we conjectured that set theoretic analogues of the results in this paper are also valid. At the time of going to print, it was brought to the attention of the author that a generalized set theoretic Lemme de Comparaison was proved in [6].

2. Preliminaries on sheaf theory

Definition 2.1. Consider a small pre-additive category $\mathfrak{u}$. An (additive Grothendieck) topology $\mathcal{T}$ on $\mathfrak{u}$ is given by specifying for every object U in $\mathfrak{u}$ a collection $\mathcal{T}(U)$ of subfunctors of $\mathfrak{u}(-,U)$ in $\mathbf{Pr}(\mathfrak{u})$ satisfying the following axioms:

(T1) $\mathfrak{u}(-,U)\in\mathcal{T}(U)$;

(T2) for $R\in\mathcal{T}(U)$ and $f:V\to U$ in $\mathfrak{u}$, the pullback $f^{-1}R$ in $\mathbf{Pr}(\mathfrak{u})$ of R along $f:\mathfrak{u}(-,V)\to\mathfrak{u}(-,U)$ is in $\mathcal{T}(V)$;

(T3) consider $S\in\mathcal{T}(U)$ and an arbitrary subfunctor R of $\mathfrak{u}(-,U)$; if for every V in $\mathfrak{u}$ and for every $f\in S(V)$ the pullback $f^{-1}R$ is in $\mathcal{T}(V)$, it follows that R is in $\mathcal{T}(U)$.

Convention 2.2. For a subfunctor $R\to\mathfrak{u}(-,U)$ we will always assume that $R(V)$ is a subgroup of $\mathfrak{u}(V,U)$. This allows us to make no notational distinction between the subfunctor R and the settheoretic union R of all the sets $R(V)$ for V in $\mathfrak{u}$.

Definition 2.3. The subfunctor $R_{\mathcal{F}}$ *generated by* a collection $\mathcal{F}$ of maps $(f_i:U_i\to U)_i$ is the smallest subfunctor of $\mathfrak{u}(-,U)$ containing $\mathcal{F}$. This subfunctor consists of all finite sums of maps that factor through an f_i in $\mathcal{F}$. The collection $\mathcal{F}$ is called a *covering* with respect to a topology $\mathcal{T}$ if $R_{\mathcal{F}}$ is an element of $\mathcal{T}(U)$.

Remark 2.4. If $\mathfrak{u}$ is the one-object category associated to a ring A, the axioms in Definition 2.1 correspond to those of a Gabriel topology on A [4].

Definition 2.5. Consider a small pre-additive category $\mathfrak{u}$ with a topology $\mathcal{T}$. A presheaf F in $\mathbf{Pr}(\mathfrak{u})$ is called a *sheaf* (resp. a *separated presheaf*) if for every $r:R\to\mathfrak{u}(-,U)$

in $\mathscr{T}$ and for every natural transformation $\alpha : R \to F$, there is a unique (resp. there is at most one) $\beta : \mathfrak{u}(-, U) \to F$ with $\beta r = \alpha$.

We will denote by $\mathbf{Sh}(\mathfrak{u}, \mathscr{T})$ the full subcategory of $\mathbf{Pr}(\mathfrak{u})$ with as objects precisely the sheaves.

The following Theorem can be extracted from [3].

Theorem 2.6. *For every additive topology $\mathscr{T}$ on $\mathfrak{u}$, $\mathbf{Sh}(\mathfrak{u}, \mathscr{T}) \to \mathbf{Pr}(\mathfrak{u})$ is a localization (whose exact left adjoint is called the sheafication functor). Conversely, if $i : \mathscr{L} \to \mathbf{Pr}(\mathfrak{u})$ is a localization with an exact left adjoint a, there is a unique topology $\mathscr{T}$ on $\mathfrak{u}$ such that the essential image of i is $\mathbf{Sh}(\mathfrak{u}, \mathscr{T})$. This topology is such that a subfunctor $r : R \to \mathfrak{u}(-, U)$ is in $\mathscr{T}$ if and only if $a(r)$ is an $\mathscr{L}$-isomorphism.*

For more details on sheaves over enriched categories, we refer the reader to [3].

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. Throughout we fix an additive functor

$$u : \mathfrak{u} \to \mathscr{C}$$

from a small pre-additive category $\mathfrak{u}$ to a Grothendieck category $\mathscr{C}$. There is a unique colimit preserving functor $\upsilon : \mathbf{Pr}(\mathfrak{u}) \to \mathscr{C}$ extending u with a right adjoint

$$\imath : \mathscr{C} \to \mathbf{Pr}(\mathfrak{u}) : C \mapsto \mathscr{C}(u(-), C).$$

For U in $\mathfrak{u}$, the canonical adjunction arrow $\eta_U : \mathfrak{u}(-, U) \to \imath(\upsilon(\mathfrak{u}(-, U)))$ is given by

$$(\eta_U)_V : \mathfrak{u}(V, U) \to \mathscr{C}(u(V), u(U)) : f \mapsto u(f).$$

We will now refine some definitions occurring in the introduction.

Definition 3.1. In an arbitrary category $\mathscr{C}$, consider a collection of maps $c_i : C_i \to C$ for $i \in I$. The collection is called *epimorphic* if $cc_i = dc_i$ for every $i \in I$ implies $c = d$. We will call a collection $(f_i : U_i \to U)_i$ of $\mathfrak{u}$-maps *epimorphic* (*with respect to u*) if the collection $(u(f_i))_i$ is epimorphic in $\mathscr{C}$. We will call a subfunctor $R \to \mathfrak{u}(-, U)$ epimorphic if the corresponding collection of maps $V \to U$ is epimorphic (see Convention 2.2). Analogously, we will call a subfunctor $R \to \mathscr{C}(u(-), C)$ epimorphic if the corresponding collection of maps $u(V) \to C$ is epimorphic.

The definition of (G), (F) and (FF) in the introduction may be reformulated in a way that is more convenient for the sequel.

- (G) For a $\mathscr{C}$-object C, consider the collection R_C of $\mathscr{C}$-maps $u(U) \to C$ with U in $\mathfrak{u}$. The functor u satisfies (G) if and only if for every $\mathscr{C}$-object C, R_C is epimorphic.
- (F) For a $\mathscr{C}$-map $c : u(U) \to u(V)$, consider the collection R_c of $\mathfrak{u}$-maps $f : U_f \to U$ with $cu(f) = u(g)$ for some $\mathfrak{u}$-map g. The functor u satisfies (F) if and only if for every $\mathscr{C}$-map c, R_c is epimorphic.

(FF) For a $\mathfrak{u}$-map $f: U \to V$, consider the collection R_f of $\mathfrak{u}$-maps $g: U_g \to U$ with $fg = 0$. The functor u satisfies (FF) if and only if $u(f) = 0$ implies that R_f is epimorphic.

Properties (G) and (F) may be combined more economically in one statement.

(GF) For a finite collection $(c_i)_{i\in I}$ of $\mathscr{C}$-maps $c_i: C \to u(U_i)$, consider the collection R_I of all maps $c: u(U_c) \to C$ such that for every $i \in I$, $c_i c = u(f_i)$ for some f_i in $\mathfrak{u}$. We say that u satisfies (GF) if for every such collection $(c_i)_{i\in I}$, R_I is epimorphic.

Lemma 3.2. (GF) $\Leftrightarrow$ (G) *and* (F).

Proof. That (G) and (F) implies (GF) is shown by induction on $|I|$. Let $|I| = 0$ and consider C in $\mathscr{C}$. By (G), the collection $R_\varnothing$ of all maps $c: u(U_c) \to C$ is epimorphic. Now consider $(c_i)_{i\in I}$ as in (GF) and suppose, by induction, that $R_{I\setminus\{j\}}$ is epimorphic. Then for every $c: u(U_c) \to C$ in $R_{I\setminus\{j\}}$, by (F), the collection $R_{c_j c}$ of maps $f: U_{(c,f)} \to U_c$ with $c_j c u(f) = u(g_{(c,f)})$ is epimorphic. Hence the collection of compositions $cu(f)$ is epimorphic, and all compositions $c_i c u(f)$ for $i \in I$ can clearly be written as $u(f_i)$ for some f_i in $\mathfrak{u}$.

Conversely, suppose (GF) holds. (G) is precisely (GF) for $|I| = 0$. To see that (F) holds, consider a $\mathscr{C}$-map $c: u(U) \to u(V)$. It suffices to apply (GF) to $(c, 1_{u(U)})$. □

3.1. *Proof of* (**1**) $\Rightarrow$ (**2**)

In this section, we suppose that $\imath$ is fully faithful and υ is exact.

Lemma 3.3. *u satisfies* (G).

Proof. Immediate since υ is the left adjoint of the faithful functor $\imath$. □

Lemma 3.4. *Consider a subfunctor* $r: R \to \mathfrak{u}(-, U)$ *in* $\mathbf{Pr}(\mathfrak{u})$. *The following are equivalent*:

(1) $\upsilon(r)$ *is a* $\mathscr{C}$*-isomorphism*;
(2) $\upsilon(r)$ *is a* $\mathscr{C}$*-epimorphism*;
(3) R *is epimorphic.*

Proof. Since υ is exact, $\upsilon(r)$ is always a $\mathscr{C}$-monomorphism. This proves the equivalence of 1 and 2. For the equivalence of 2 and 3, it suffices to write r as the image of the map $r': \coprod_{f\in R} \mathfrak{u}(-, U_f) \to \mathfrak{u}(-, U)$ induced by the maps $f: U_f \to U$ in R, and note that $\upsilon(r')$ is induced by the maps $u(f)$ for $f \in R$. □

Lemma 3.5. *u satisfies* (F).

Proof. Consider $c : u(U) \to u(V)$ in $\mathscr{C}$. Consider the pullback

$$\begin{array}{ccccc} P & \longrightarrow & & & \mathfrak{u}(-,V) \\ {\scriptstyle p}\downarrow & & & & \downarrow{\scriptstyle \eta_V} \\ \mathfrak{u}(-,U) & \xrightarrow[\eta_U]{} & \mathscr{C}(u(-),u(U)) & \xrightarrow[\imath(c)]{} & \mathscr{C}(u(-),u(V)) \end{array}$$

in $\mathbf{Pr}(\mathfrak{u})$ in which η_U and η_V are the canonical adjunction arrows. Let $r : R \to \mathfrak{u}(-,U)$ denote the image of p. Since $\imath$ is fully faithful, $\upsilon(\eta_V)$ is an isomorphism. Hence since υ is exact, $\upsilon(r)$ is an isomorphism, meaning precisely that the collection R_c of (F) is epimorphic. □

Lemma 3.6. *u satisfies* (FF).

Proof. Consider $f : U \to V$ in $\mathfrak{u}$ and let $k : K \to \mathfrak{u}(-,U)$ be the kernel of the corresponding $f : \mathfrak{u}(-,U) \to \mathfrak{u}(-,V)$ in $\mathbf{Pr}(\mathfrak{u})$. Since υ is exact, $u(f) = 0$ implies that $\upsilon(k)$ is an isomorphism, meaning precisely that the collection R_f of (FF) is epimorphic. □

3.2. Proof of $(\mathbf{2}) \Rightarrow (\mathbf{1})$

We will actually prove the following more precise result.

Theorem 3.7. *Suppose that u satisfies conditions* (G), (F) *and* (FF). *Let $\mathscr{T}$ be given by the epimorphic subfunctors of representable functors on $\mathfrak{u}$. Then $\mathscr{T}$ is a topology on $\mathfrak{u}$ and $\imath : \mathscr{C} \to \mathbf{Pr}(\mathfrak{u}) : C \mapsto \mathscr{C}(u(-),C)$ factors over an equivalence $\varsigma : \mathscr{C} \to \mathbf{Sh}(\mathfrak{u},\mathscr{T})$. The sheafication functor $a : \mathbf{Pr}(\mathfrak{u}) \to \mathbf{Sh}(\mathfrak{u},\mathscr{T})$ is given by $\mathfrak{u}(-,U) \mapsto \mathscr{C}(u(-),u(U))$.*

The last part of this theorem immediately implies the following corollary:

Corollary 3.8. (1) *The following are equivalent*:

(a) *u is faithful*;
(b) *the functors $\mathfrak{u}(-,U)$ for U in $\mathfrak{u}$ are separated presheaves.*

(2) *The following are equivalent*:

(a) *u is fully faithful*;
(b) *the functors $\mathfrak{u}(-,U)$ for U in $\mathfrak{u}$ are sheaves.*

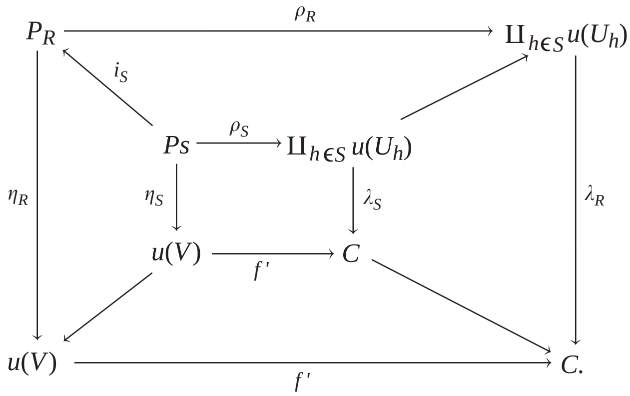
We now start with the actual proof. The following lemma uses (FF) only in part 2.

Lemma 3.9. (1) *Consider an epimorphic subfunctor $R \to \mathscr{C}(u(-),C)$ and a $\mathscr{C}$-map $f' : u(V) \to C$. The collection $f'^{-1}R$ of all $\mathfrak{u}$-maps $h : V_h \to V$ with $f'u(h) \in R(V_h)$ defines an epimorphic subfunctor of $\mathfrak{u}(-,V)$.*

(2) *Consider an epimorphic subfunctor R of* $\mathfrak{u}(-,U)$ *and a* $\mathfrak{u}$-map $f:V\to U$. *The collection* $f^{-1}R$ *of all* $\mathfrak{u}$-*maps* $h:V_h\to V$ *with* $fh\in R$ *defines an epimorphic subfunctor of* $\mathfrak{u}(-,V)$.

Proof. The proofs of 1 and 2 are very similar so we give them simultaneously. For 2, put $C=u(U)$ and $f'=u(f)$. Let R also denote the collection of couples $h=(h',U_h)$ with $h':u(U_h)\to C$ in $R(U_h)$.

For every subset $S\subset R$, consider the canonical map $\lambda_S:\coprod_{h\in S}u(U_h)\to C$ and consider its pullback $\eta_S:P_S\to u(V)$ along f'. Since $\coprod_{h\in R}u(U_h)$ is the filtered colimit of all $\coprod_{h\in S}u(U_h)$ where S is finite, and since filtered colimits are exact in $\mathscr{C}$, we obtain the following diagram in which P_R is the filtered colimit of the objects P_S where S is finite:



For every finite $S\subset R$, let pr_h and in_h denote the canonical projections and injections of the biproduct $\coprod_{h\in S}u(U_h)$. Let R_S denote the collection of maps $c:u(U_c)\to P_S$ for which $\eta_S c$ and all the maps $\mathrm{pr}_h\rho_S c$ for $h\in S$ can be written as $u(g)$ for some $g\in\mathfrak{u}$. Let T denote the collection of $\mathfrak{u}$-maps g with $u(g)=\eta_S c$ for some finite S and $c\in R_S$. By (GF), every R_S is epimorphic. Since λ_R and hence also η_R in an epimorphism, it follows that T is epimorphic. For $g\in T$ with $u(g)=\eta_S c$ and $\mathrm{pr}_h\rho_S c=u(f_h)$ for $h\in S$, we can compute

$$f'u(g)=f'\eta_S c=\lambda_S\left(\sum_{h\in S}\mathrm{in}_h\,\mathrm{pr}_h\right)\rho_S c.$$

First, we will finish case 1. Since

$$\lambda_S\left(\sum_{h\in S}\mathrm{in}_h\,\mathrm{pr}_h\right)\rho_S c=\sum_{h\in S}h'u(f_h),$$

T is an epimorphic part of $f'^{-1}R$.

Next, we will finish case 2. In this case,

$$u(fg)=f'u(g)=\lambda_S\left(\sum_{h\in S}\mathrm{in}_h\,\mathrm{pr}_h\right)\rho_S c=u\left(\sum_{h\in S}hf_h\right).$$

By (FF), the collection T_g of $\mathfrak{u}$-maps $k : V_k \to U_c$ with $fgk = \sum_{h \in S} h f_h k$ is epimorphic. Finally, the collection T' of all compositions gk with $g \in T$ and $k \in T_g$ is an epimorphic part of $f^{-1}R$. □

Corollary 3.10. *The epimorphic subfunctors define an additive topology* $\mathscr{T}$ *on* $\mathfrak{u}$.

Proof. Conditions (T1) and (T3) are trivially verified, and (T2) is precisely Lemma 3.9(2). □

As in Section 2, we will denote by $i : \mathbf{Sh}(\mathfrak{u}, \mathscr{T}) \to \mathbf{Pr}(\mathfrak{u})$ the canonical inclusion and by $a : \mathbf{Pr}(\mathfrak{u}) \to \mathbf{Sh}(\mathfrak{u}, \mathscr{T})$ an exact left adjoint.

Corollary 3.11. *If* $r : R \to \mathscr{C}(u(-), C)$ *is an epimorphic subfunctor, then* $a(r)$ *is an isomorphism.*

Proof. Consider the pullback

$$\begin{array}{ccc} R & \xrightarrow{\ r\ } & U(u(-),C) \\ \uparrow & & \uparrow \omega \\ P & \xrightarrow[\ p\ ]{} & u(-,U) \end{array}$$

for an arbitrary ω. Lemma 3.9(1) states that p is epimorphic, hence a covering of U. So clearly every functor $\alpha : \mathscr{C}(u(-), C) \to F$ to a sheaf F with $\alpha r = 0$ satisfies $\alpha = 0$. □

The following lemma uses (FF) only in part 1.

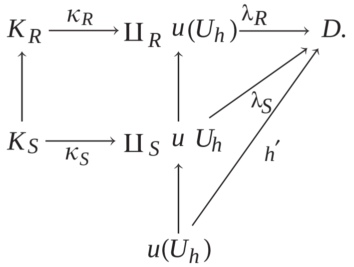
Lemma 3.12. *Consider* $\imath : \mathscr{C} \to \mathbf{Pr}(\mathfrak{u}) : C \mapsto \mathscr{C}(u(-), C)$. *The following hold*:

(1) $\imath$ *is fully faithful.*
(2) *For every* $C \in \mathscr{C}$, $\imath(C)$ *is a sheaf on* $(\mathfrak{u}, \mathscr{T})$.

Proof. The proofs of 1 and 2 are very similar so we give them simultaneously. For 1, consider $\alpha : \mathscr{C}(u(-), D) \to \mathscr{C}(u(-), C)$. We will show that α is the image of a unique $b : D \to C$. This proves fully faithfulness of $\imath$. Let R denote the collection of all couples $h = (h', U_h)$ with $h' : u(U_h) \to D$ a $\mathscr{C}$-map and put $\alpha_{U_h}(h) = \alpha_{U_h}(h')$.

For 2, consider an epimorphic subfunctor $r : R \to \mathfrak{u}(-, U)$ and $\alpha : R \to \mathscr{C}(u(-), C)$. Put $u(U) = D$. For $h : U_h \to U$ in R, put $h' = u(h)$.

In both cases, consider for every subset $S \subset R$ the map $\lambda_S : \coprod_S u(U_h) \to D$ determined by the maps h' for $h \in S$, and consider the kernel $\kappa_S : K_S \to \coprod_S u(U_h)$ of λ_S. Since $\coprod_R u(U_h)$ is the filtered colimit of all $\coprod_S u(U_h)$ where S is finite, and since filtered colimits are exact in $\mathscr{C}$, we obtain the following diagram in which K_R is the filtered colimit of the objects K_S where S is finite:



For every $S \subset R$, the maps $\alpha_{U_h}(h): u(U_h) \to C$ determine a canonical map $\mu_S : \coprod_S u(U_h) :\to C$. Suppose $\mu_R \kappa_R = 0$. Since λ_R is the cokernel of κ_R, we obtain a unique morphism $b: D \to C$ with $bh' = \alpha_{U_h}(h)$ for every $h \in R$. So in both cases it remains to show that $\mu_R \kappa_R = 0$, or equivalently that $\mu_S \kappa_S = 0$ for every finite S. In order to do this, let in_h and pr_h denote the canonical injections and projections of the biproduct $\coprod_S u(U_h)$ and consider the collection R_S of all maps $c: u(V_c) \to K_S$ such that for every $h \in S$, $\mathrm{pr}_h \kappa_S c = u(f_h)$ for some f_h in $\mathfrak{u}$. By (GF), R_S is epimorphic.

First, we will finish case 1. For every $c \in R_S$ we have that

$$\mu_S \kappa_S c = \sum_{h \in S} \alpha_{U_h}(h') \, \mathrm{pr}_h \kappa_S c = \alpha_{V_c}\left(\sum_{h \in S} h' u(f_h) \right) = \alpha_{V_c}(\lambda_S \kappa_S c) = 0,$$

hence it follows that $\mu_S \kappa_S = 0$.

Next, we will finish case 2. For every $c \in R_S$ we have that

$$\mu_S \kappa_S c = \sum_{h \in S} \alpha_{U_h}(h) \, \mathrm{pr}_h \kappa_S c = \alpha_{V_c}\left(\sum_{h \in S} h f_h \right).$$

Now

$$u\left(\sum_{h \in S} h f_h \right) = \lambda_S \kappa_S c = 0.$$

So by (FF), the collection $R_{(S,c)}$ of maps $g: W_g \to V_c$ for which $\sum_{h \in S} h f_h g = 0$ is epimorphic. For every $g \in R_{(S,c)}$, we have that

$$\alpha_{V_c}\left(\sum_{h \in S} h f_h \right) u(g) = \alpha_{W_g}\left(\sum_{h \in S} h f_h g \right) = 0.$$

Hence it follows that $\alpha_{V_c}(\sum_{h \in S} h f_h) = 0$ and consequently $\mu_S \kappa_S = 0$. □

Lemma 3.13. *Consider a natural transformation $\alpha: \mathfrak{u}(-, U) \to F$ to a sheaf F and suppose $f: V \to U$ in $\mathfrak{u}$ is such that the collection R_f of* (FF) *is epimorphic. Then $\alpha_V(f) = 0$.*

Proof. The collection R_f of maps $g: V_g \to V$ with $fg = 0$ is a covering of V. The composition $\alpha f: \mathfrak{u}(-, V) \to F$ is zero on R_f, proving that $\alpha_V(f) = (\alpha f)(1_V) = 0$. □

Lemma 3.14. *For* $U \in \mathfrak{u}$, *consider* $\eta_U : \mathfrak{u}(-,U) \to \mathscr{C}(u(-),u(U))$ *in* $\mathbf{Pr}(\mathfrak{u})$. $(\mathscr{C}(u(-),u(U)),\eta_U)$ *is a reflection of* $\mathfrak{u}(-,U)$ *along* $i:\mathbf{Sh}(\mathfrak{u},\mathscr{T}) \to \mathbf{Pr}(\mathfrak{u})$.

Proof. Consider a sheaf F and a natural transformation $\alpha : \mathfrak{u}(-,U) \to F$. We have to define a unique $\beta : \mathscr{C}(u(-),u(U)) \to F$ with $\beta\eta = \alpha$. Consider $c : u(V) \to u(U)$. By (F), the collection R_c of maps $f : V_f \to V$ with $cu(f) = u(g_f)$ is epimorphic. Consider the cover $R_c \to \mathfrak{u}(-,V)$. By Lemma 3.13 and (FF), we can define a natural transformation $\mu : R_c \to F$ by putting $\mu_{V_f}(f) = \alpha_{V_f}(g_f)$. Since F is a sheaf, μ corresponds to a unique element $\beta_V(c) \in F(V)$. The β defined in this way is natural and unique with $\beta\eta = \alpha$. □

By Lemma 3.12, there is a fully faithful functor $\varsigma : \mathscr{C} \to \mathbf{Sh}(\mathfrak{u},\mathscr{T})$ with $i\varsigma = \imath$. Since $\imath$ is a right adjoint and i is a fully faithful right adjoint, ς is limit preserving. We set out to prove that ς is an equivalence of categories.

Lemma 3.15. ς *preserves coproducts.*

Proof. Consider a collection of $\mathscr{C}$-objects $(C_i)_{i \in I}$. There is a canonical morphism

$$\gamma : \coprod \mathscr{C}(u(-),C_i) \to \mathscr{C}\left(u(-),\coprod C_i\right)$$

in $\mathbf{Pr}(\mathfrak{u})$ of which we have to prove that the associated morphism

$$a(\gamma) : \coprod \varsigma(C_i) \to \varsigma\left(\coprod C_i\right)$$

is an isomorphism in $\mathbf{Sh}(\mathfrak{u},\mathscr{T})$. Since γ is obviously a monomorphism in $\mathbf{Pr}(\mathfrak{u})$, it suffices by Corollary 3.11 that the image of γ is epimorphic, which is clearly the case. □

Lemma 3.16. ς *is exact.*

Proof. Consider a $\mathscr{C}$-epimorphism $c : C \to C'$ and the corresponding

$$\gamma : \mathscr{C}(u(-),C) \to \mathscr{C}(u(-),C')$$

in $\mathbf{Pr}(\mathfrak{u})$. Since the image of γ is clearly epimorphic, it follows from Corollary 3.11 that $\varsigma(c) = a(\gamma)$ is an epimorphism. □

Lemma 3.17. ς *is an equivalence of categories.*

Proof. It remains to show that ς is essentially surjective. Consider a sheaf F in $\mathbf{Pr}(\mathfrak{u})$. By Lemma 3.14, the objects $\mathscr{C}(u(-),u(U))$ for $U \in \mathfrak{u}$ generate $\mathbf{Sh}(\mathfrak{u},\mathscr{T})$. Hence, by Lemma 3.15, we can find C and C' in $\mathscr{C}$ and an exact sequence $\varsigma(C') \to \varsigma(C) \to F \to 0$ in $\mathbf{Sh}(\mathfrak{u},\mathscr{T})$. If we let C'' denote the cokernel of the corresponding map $C' \to C$, it follows from Lemma 3.16 that $F \cong \varsigma(C'')$. □

This finishes the proofs of Theorems 3.7 and 1.2.

4. Lemme de comparaison

In this section we apply our main result Theorem 1.2 to give a generalization (in the additive setting) of the "Lemme the comparaison" [1] which gives sufficient conditions for a functor between small categories to yield an equivalence on the level of sheaf categories. For a set theoretical Lemme de Comparaison in the same generality we refer the reader to [6].

We start with a definition which abstracts the behaviour of epimorphic collections. This definition is inspired by the notion of a "pretopology" in [1].

Definition 4.1. Assume $\mathscr{C}$ is an arbitrary category. A *covering system* $\mathscr{R}$ on $\mathscr{C}$ is given by specifying for every C in $\mathscr{C}$ a set $\mathscr{R}(C)$ of coverings of C. A *covering* of C is by definition a collection of maps $C_i \to C$ in $\mathscr{C}$. These coverings have to satisfy the following transitivity property: if $(C_i \to C)_i$ is a covering of C and if for every i, $(C_{ij} \to C_i)_j$ is a covering of C_i then the collection of compositions $(C_{ij} \to C_i \to C)_{ij}$ is a covering of C.

If $\mathfrak{u}$ is a small pre-additive category equipped with a topology $\mathscr{T}$ then $\mathfrak{u}$ has an associated covering system $\mathscr{R}_{\mathscr{T}}$ where the coverings are as in Definition 2.3. Conversely, we will show that every "reasonable" covering system $\mathscr{R}$ on $\mathfrak{u}$ naturally determines a localization of $\mathbf{Pr}(\mathfrak{u})$ (and hence a topology $\mathscr{T}_{\mathscr{R}}$ on $\mathfrak{u}$). This correspondence is such that $\mathscr{T}_{\mathscr{R}_{\mathscr{T}}} = \mathscr{T}$.

Consider a small pre-additive category $\mathfrak{u}$ endowed with a covering system $\mathscr{R}$ with $\mathscr{R}(U) \neq \varnothing$ for all U in $\mathfrak{u}$. Let $\mathscr{N}(\mathfrak{u},\mathscr{R})$ be the following full subcategory of $\mathbf{Pr}(\mathfrak{u})$: a presheaf F belongs to $\mathscr{N}(\mathfrak{u},\mathscr{R})$ if for every $x \in F(U)$, there exists a covering $f_i : U_i \to U$ in $\mathscr{R}$ such that for every i, $F(f_i)(x) = 0$. Put $\mathscr{L}(\mathfrak{u},\mathscr{R}) = \mathscr{N}(\mathfrak{u},\mathscr{R})^{\perp}$, i.e. $\mathscr{L}(\mathfrak{u},\mathscr{R})$ is the full subcategory of $\mathbf{Pr}(\mathfrak{u})$ containing all presheaves F with $\mathrm{Hom}_{\mathbf{Pr}(\mathfrak{u})}(N,F) = 0 = \mathrm{Ext}^1_{\mathbf{Pr}(\mathfrak{u})}(N,F)$ for all N in $\mathscr{N}(\mathfrak{u},\mathscr{R})$.

We have the following

Proposition 4.2. *Let $\mathfrak{u}$ be a small pre-additive category endowed with a covering system $\mathscr{R}$ with $\mathscr{R}(U) \neq \varnothing$ for all U in $\mathfrak{u}$. There is a topology $\mathscr{T}_{\mathscr{R}}$ on $\mathfrak{u}$ with*

$$\mathscr{L}(\mathfrak{u},\mathscr{R}) = \mathbf{Sh}(\mathfrak{u},\mathscr{T}_{\mathscr{R}}).$$

This topology is such that $r : R \to \mathfrak{u}(-,U)$ is in $\mathscr{T}_{\mathscr{R}}(U)$ if and only if for every $f : V \to U$ in $\mathfrak{u}$, the pullback $p : P \to \mathfrak{u}(-,V)$ of R along f contains a covering in $\mathscr{R}$.

Proof. It is easily verified that $\mathscr{N}(\mathfrak{u},\mathscr{R})$ is a localizing subcategory of $\mathbf{Pr}(\mathfrak{u})$ (i.e. $\mathscr{N}(\mathfrak{u},\mathscr{R})$ is closed under coproducts, subquotients and extensions, see for example [9]. Note that since $\mathscr{R}(U) \neq \varnothing$, $0 = \coprod_{\varnothing}$ is in $\mathscr{N}(\mathfrak{u},\mathscr{R})$.) Consequently, $\mathscr{L}(\mathfrak{u},\mathscr{R}) \to \mathbf{Pr}(\mathfrak{u})$ is a localization with an exact left adjoint a, hence by Theorem 2.6, $\mathscr{L}(\mathfrak{u},\mathscr{R}) = \mathbf{Sh}(\mathfrak{u},\mathscr{T}_{\mathscr{R}})$ for a topology $\mathscr{T}_{\mathscr{R}}$. Moreover, $r : R \to \mathfrak{u}(-,U)$ is in $\mathscr{T}_{\mathscr{R}}(U)$ if and only if $a(r)$ is an epimorphism in $\mathscr{L}(\mathfrak{u},\mathscr{R})$. By the theory of localizing subcategories, this happens precisely if $\mathrm{Coker}(r)$ is in $\mathscr{N}(\mathfrak{u},\mathscr{R})$, that is if for every $f \in \mathfrak{u}(V,U)$, there exists a covering $f_i : V_i \to V$ in $\mathscr{R}$ with $f f_i \in R(V_i)$ for every i. □

If $\mathscr{C}$ is an arbitrary category then the *epi-covering system* is given by the epimorphic collections of maps.

Covering systems can be induced along a functor.

Definition 4.3. Consider a functor $u : \mathscr{U} \to \mathscr{C}$ between pre-additive categories, such that $\mathscr{C}$ is equipped with a covering system $\mathscr{R}$. Then $(f_i : U_i \to U)_i$ is a covering for the *induced covering system* $\mathscr{R}_u$ if $(u(f_i))_i$ is a covering for $\mathscr{R}$.

We can now formulate "relative" versions of definitions (G), (F) and (FF) given in the introduction. Let $u : \mathscr{U} \to \mathscr{C}$ and $\mathscr{R}$ be as in the above definition.

- (G) We say that $(u,\mathscr{R})$ satisfies (G) if every object C in $\mathscr{C}$ has a covering of the form $(u(U_i) \to C)_i$ with U_i in $\mathscr{U}$.
- (F) We say that $(u,\mathscr{R})$ satisfies (F) if for every map $c : u(U) \to u(V)$ in $\mathscr{C}$ there exists a covering $(f_i : U_i \to U)_i$ in $\mathscr{R}_u$ such that $cu(f_i)$ is in the image of u for all i.
- (FF) We say that $(u,\mathscr{R})$ satisfies (FF) if for every map $f : U \to V$ in the kernel of u there exists a covering $(f_i : U_i \to U)_i$ in $\mathscr{R}_u$ such that $f f_i = 0$ for all i.

If $\mathfrak{u}$ is small and $\mathscr{C}$ is a Grothendieck category equipped with the epi-covering system $\mathscr{R}$ then this definition reduces to the one in the introduction. In this case, if $(u,\mathscr{R})$ satisfies (G), (F) and (FF), $\mathscr{R}_u$ is the covering system $\mathscr{R}_{\mathscr{T}}$ associated to the topology $\mathscr{T}$ on $\mathfrak{u}$ in Theorem 3.7.

Proposition 4.4. *Consider a composition of additive functors*

$$\mathscr{V} \underset{v}{\to} U \underset{u}{\to} C$$

and a covering system $\mathscr{R}$ on $\mathscr{C}$. The following is true.

1. $\mathscr{R}_{uv} = (\mathscr{R}_u)_v$;
2. *Assume that* $(u,\mathscr{R})$ *and* $(v,\mathscr{R}_u)$ *satisfy* (G), (F) *and* (FF). *Then the same holds for* $(uv,\mathscr{R})$.
3. *Assume that u is fully faithful and that* $(uv,\mathscr{R})$ *satisfies* (G), (F) *and* (FF). *Then the same holds for* $(v,\mathscr{R}_u)$.

Proof. This is straightforward from the definitions. □

Corollary 4.5. *Consider an additive functor $v : \mathfrak{v} \to \mathfrak{u}$ between small pre-additive categories and an additive topology $\mathscr{T}$ on $\mathfrak{u}$. Denote the associated covering system by the same symbol.*

(1) *Suppose* $(v, \mathscr{T})$ *satisfies* (G), (F) *and* (FF). *Then*
  (a) $\mathscr{T}_v$ *defines an additive topology on* $\mathfrak{v}$ (*the subfunctors generated by coverings in* $\mathscr{T}_v$).
  (b) *Consider* $v^*: \mathbf{Pr}(\mathfrak{u}) \to \mathbf{Pr}(\mathfrak{v}): H \mapsto Hv$. *There is a commutative square of functors*

$$\begin{array}{ccc} \mathbf{Pr}(\mathfrak{v}) & \xleftarrow{\;v^*\;} & \mathbf{Pr}(\mathfrak{u}) \\ \uparrow{\scriptstyle i'} & & \uparrow{\scriptstyle i} \\ \mathbf{Sh}(\mathfrak{v}, \mathscr{T}_v) & \xleftarrow[\;\varsigma\;]{} & \mathbf{Sh}(\mathfrak{u}, \mathscr{T}) \end{array}$$

  in which ς *is an equivalence of categories.*

(2) *Suppose there is a commutative square of functors as in* 1(b) *for an additive topology* $\mathscr{T}_v$ *on* $\mathfrak{v}$. *Then* $\mathscr{T}_v$ *is induced by* $\mathscr{T}$.

(3) *Suppose the functors* $\mathfrak{u}(-, U)$ *for* U *in* $\mathfrak{u}$ *are sheaves with respect to* $\mathscr{T}$ *and suppose there is a commutative square of functors as in* 1(b). *Then* $(v, \mathscr{T})$ *satisfies* (G), (F) *and* (FF).

Proof. If we consider v, the canonical functor $u: \mathfrak{u} \to \mathbf{Pr}(\mathfrak{u}) \to \mathbf{Sh}(\mathfrak{u}, \mathscr{T})$, and the composition uv, the result follows from Proposition 2.6, Theorems 1.2, Theorem 3.7, Corollary 3.8 and Proposition 4.4. □

5. An illustrative example

In this section we discuss a simple example which illustrates conditions (G), (F) and (FF).

Let X be a topological space and let $\mathcal{O}_X$ be a sheaf of rings on X. We denote by $\mathbf{Mod}(\mathcal{O}_X)$ and $\mathbf{PMod}(\mathcal{O}_X)$ the categories of sheaves and presheaves of $\mathcal{O}_X$-modules. $\mathbf{Mod}(\mathcal{O}_X)$ is a localization of $\mathbf{PMod}(\mathcal{O}_X)$, the exact left adjoint a being given by sheafication.

For an open $U \subset X$ let $j_U: U \to X$ be the inclusion map and let $P_U = j^p_{U,!}\mathcal{O}_U$ and $S_U = j_{U,!}\mathcal{O}_U = a(j^p_{U,!}\mathcal{O}_U)$ be the extensions by zero of $\mathcal{O}_U$ in the categories of presheaves and sheaves. We have

$$\mathrm{Hom}(P_U, P_V) = \begin{cases} \Gamma(U, \mathcal{O}_U) & \text{if } U \subset V, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and

$$\mathrm{Hom}(S_U, S_V) = \{ f \in \Gamma(U, \mathcal{O}_U) \mid f \text{ is zero on a neighborhood of } U \setminus V \}. \tag{2}$$

Thus in particular if $U \subset V$,

$$\mathrm{Hom}(P_U, P_V) = \mathrm{Hom}(S_U, S_V). \tag{3}$$

The objects $(P_U)_U$ are finitely generated projective generators of $\mathbf{PMod}(\mathcal{O}_X)$ and hence the objects $(S_U)_U$ are generators of $\mathbf{Mod}(\mathcal{O}_X)$. Let $\mathfrak{u}$ and $\mathfrak{u}'$ be the full subcategories of $\mathbf{PMod}(\mathcal{O}_X)$ and $\mathbf{Mod}(\mathcal{O}_X)$ spanned by these objects. We deduce that

$$\mathbf{PMod}(\mathcal{O}_X) \cong \mathbf{Pr}(\mathfrak{u}).$$

So $\mathbf{Mod}(\mathcal{O}_X)$ is a localization of $\mathbf{Pr}(\mathfrak{u})$. However this is *not* a consequence of the Gabriel–Popescu theorem. Indeed the Gabriel–Popescu theorem would realize $\mathbf{Mod}(\mathcal{O}_X)$ as a localization of $\mathbf{Pr}(\mathfrak{u}')$ and as seen from (1) and (2) the relation between $\mathbf{Pr}(\mathfrak{u})$ and $\mathbf{Pr}(\mathfrak{u}')$ is obscure.

On the other hand, the fact that $\mathbf{Mod}(\mathcal{O}_X)$ is a localization of $\mathbf{Pr}(\mathfrak{u})$ is a consequence of Theorem 3.7. Indeed if $u : \mathfrak{u} \to \mathbf{Mod}(\mathcal{O}_X)$ is the sheafication functor then it satisfies (G), (F) and (FF). This follows of course from the converse direction of Theorem 1.2, but let us verify it directly.

We already know that (G) holds, and (FF) follows from the fact that the presheaves P_U are separated.

To prove (F) consider $f \in \mathrm{Hom}(S_U, S_V)$. We identify f with a section of $\mathcal{O}_U$ as in (2). For every $U' \subset U$ there is a canonical map $j_{U',U} : P_{U'} \to P_U$ determined by $1 \in \Gamma(U', \mathcal{O}_{U'})$. It is easily seen by looking at stalks that a collection $(j_{U_i,U})_i$ with $U = \bigcup_i U_i$ is epimorphic with respect to u. Now let $W \subset U$ be a neighborhood of $U \setminus V$ on which f is zero. Thus $(U \cap V) \cup W = U$ hence $(j_{U\cap V,U}, j_{W,U})$ is epimorphic.

By (3) the composition $fu(j_{U\cap V,U}) : S_{U\cap V} \to S_U \to S_V$ is in the image of $\mathrm{Hom}(P_{U\cap V}, P_V)$ and clearly the composition $fu(j_{W,U}) : S_W \to S_U \to S_V$ is zero, so it is also in the image of $\mathrm{Hom}(P_W, P_V)$. This proves (F).

To finish this example let us give an application of Corollary 4.5. Let $\mathscr{B}$ be a basis for the topology on X. Let $\mathfrak{v}$ be the full subcategory of $\mathbf{PMod}(\mathcal{O}_X)$ spanned by the objects P_U for $U \in \mathscr{B}$. Let $\mathscr{T}$ be the topology of Theorem 3.7 that yields an equivalence of categories $\mathbf{Mod}(\mathcal{O}_X) \cong \mathbf{Sh}(\mathfrak{u}, \mathscr{T})$. The inclusion $v : \mathfrak{v} \to \mathfrak{u}$ is fully faithful and clearly satisfies (G). Hence by Corollary 4.5 there is an equivalence of categories $\mathbf{Sh}(\mathfrak{v}, \mathscr{T}_v) \cong \mathbf{Sh}(\mathfrak{u}, \mathscr{T})$. This corresponds of course to the fact that sheaf-categories are determined by a basis for the topology.

Acknowledgements

I wish to thank Michel Van den Bergh for some of the ideas in this paper as well as for several interesting discussions.

References

[1] M. Artin, A. Grothendieck, J.L. Verdier, Théorie des topos et cohomologie étale des schémas, in: SGA4: Séminaire de Géometrie Algébrique du Bois Marie 1963/64, Lecture Notes in Mathematics, Vol. 269, Springer, Berlin, 1972.

[2] F. Borceux, Handbook of Categorical Algebra 3, in: Encyclopedia of Mathematics and its Applications, Vol. 52, Cambridge University Press, Cambridge, 1994.

[3] F. Borceux, C. Quinteiro, A theory of enriched sheaves, Cah. Top. Géom. Diff. Catég. 2 (1996) 145–162.

[4] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France 90 (1962) 323–448.
[5] P. Gabriel, N. Popescu, Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes, C. R. Acad. Sci. Paris 258 (1964) 4188–4191.
[6] A. Kock, I. Moerdijk, Presentations of étendues. Cah. Top. Géom. Diff. Catég. (1991), no. XXXII-2.
[7] W. Lowen, M. Van den Bergh, Deformation theory of abelian categories, 2003, submitted.
[8] B. Mitchell, A quick proof of the Gabriel–Popesco theorem, J. Pure Appl. Algebra 20 (1981) 313–315.
[9] N. Popescu, Abelian Categories with Applications to Rings and Modules, in: L.M.S. Monographs, Vol. 3, Academic Press, New York, 1973.
[10] M. Takeuchi, A simple proof of Gabriel and Popesco's theorem, J. Algebra 18 (1971) 112–113.