The \textit{qd}-scheme

For a given sequence \( \{a_n\} \), follow the \textit{qd}-scheme:

\begin{itemize}
  \item \textbf{Initialize:}
    \begin{align*}
      e_0^{(n)} &= 0, & n &= 1, 2, \ldots; \\
      q_0^{(n)} &= a_{n+1}/a_n, & n &= 0, 1, 2, \ldots.
    \end{align*}
\end{itemize}

\begin{itemize}
  \item Continue with:
    \begin{align*}
      e_m^{(n)} &= q_m^{(n+1)} - q_m^{(n)} + e_m^{(n+1)}, & m &= 1, 2, \ldots; \\
      q_m^{(n)} &= \frac{e_m^{(n)}}{e_m^{(n+1)} - q_m^{(n)}}, & n &= 0, 1, 2, \ldots.
    \end{align*}
\end{itemize}

A two-dimensional array can be formed:

\begin{align*}
  \begin{array}{cccc}
    q_0^{(0)} & q_1^{(0)} & q_2^{(0)} & q_3^{(0)} \\
    0 & e_1^{(0)} & e_2^{(0)} & e_3^{(0)} \\
    0 & q_1^{(1)} & q_2^{(1)} & q_3^{(1)} \\
    0 & q_2^{(2)} & q_3^{(2)} & \vdots
  \end{array}
\end{align*}

A more stable progressive form exists for the same \textit{qd}-scheme [3]. In addition, a breakdown free version of the \textit{qd}-algorithm is described in [1].

References


Rutishauser’s \textit{qd}-algorithm

Let the function \( f \) be analytic at \( z = 0 \) and meromorphic in the disk \( D \) with \( |z| < \delta \). That is, \( f \) is analytic on all \( D \) except a set of isolated points (poles).

Let its poles \( z_i = w_i^{-1} \) in \( D \), and let it be numbered such that

\[ |u_1| \geq |u_2| \geq \cdots \geq |u_i| > 0 = |u_{i+1}|, \]

each pole occurring as many times in the sequence \( \{u_i\} \) (or \( \{z_i\} \)) as it is indicated by its order.

Let \( \{a_n\} \) be the sequence formed by the Taylor coefficients of \( f \):

\[ f = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots. \]

Then the \textit{qd}-scheme has the following properties:

\begin{enumerate}
  \item \textbf{(a)} for each \( 0 < j \leq t \) and \( |u_j| > |u_{j+1}| \), it holds that
    \[ \lim_{n \to \infty} e_j^{(n)} = 0; \]
  \item \textbf{(b)} for each \( 0 < j < t \) and \( |u_{j-1}| > |u_j| > |u_{j+1}| \), it holds that
    \[ \lim_{n \to \infty} q_j^{(n)} = u_j = \frac{1}{z_j}; \]
  \item \textbf{(c)} for each \( j \) and \( \ell > 1 \) such that \( 0 < j < j+\ell < t \) and \( |u_{j-1}| > |u_j| = \cdots = |u_{j+\ell-1}| > |u_{j+\ell}| \), it holds that for the polynomials \( \rho_{\ell}^{(n)} \) defined by
    \[ \rho_{\ell}^{(n)}(u) = 1, \]
    \[ n \geq 0, \quad s = 0, 1, \ldots, \ell - 1, \]
    there exists a subsequence that converges to
    \[ (u - u_j) \cdots (u - u_{j+\ell-1}); \]
  \item \textbf{(d)} for \( j = t \) we have
    \[ e_t^{(n)} = 0, \quad n > 0. \]
\end{enumerate}

Sparse Multivariate Polynomial Interpolation via the Quotient-Difference Algorithm

We consider the black box interpolation of polynomial [2]

\[ p(x, y, z) = \pi x^{2/3} y^{2/3} z - \sqrt{2} x^{2/3} + 100 z^3. \]

\textbf{Compare to Rutishauser’s \textit{qd}-algorithm for equimodular poles: (c) and (d)}

Suppose the number (or an upper bound) of non-zero terms \( t \) is unknown. If the partial degree (or its upper bound) in each variable is given, choose \( \omega_1 = \exp(2\pi i/7), \omega_2 = \exp(2\pi i/11), \omega_3 = \exp(2\pi i/13) \) and follow the \textit{qd}-scheme for sequence \( \{a_n = p(\omega_1, \omega_2, \omega_3)\} \).

\textbf{Sparse polynomial interpolation via the \textit{qd}-algorithm}

We consider the black box interpolation of polynomial [2]

\[ p(x, y, z) = \pi x^{2/3} y^{2/3} z - \sqrt{2} x^{2/3} + 100 z^3. \]

\textbf{Compare to Rutishauser’s \textit{qd}-algorithm for non-equimodular poles: (a), (b), and (d)}

When neither \( t \) nor (an upper bound of) the partial degree is known, choose \( \xi_1 = 1/3, \xi_2 = 1/5, \xi_3 = 1/2 \), in which 3, 5, 2 are pairwise relatively prime, and follow the \textit{qd}-scheme for sequence \( a_n = \{p(\xi_1, \xi_2, \xi_3)\} \).

\[ q_0^{(0)} e_1^{(0)} q_2^{(0)} e_2^{(0)} q_3^{(0)} e_3^{(0)} q_4^{(0)} \]
\[ q_1^{(1)} e_1^{(1)} q_2^{(1)} e_2^{(1)} q_3^{(1)} e_3^{(1)} q_4^{(1)} \]
\[ q_2^{(2)} e_2^{(2)} q_3^{(2)} e_3^{(2)} q_4^{(2)} \]
\[ q_3^{(3)} e_3^{(3)} q_4^{(3)} \]
\[ q_4^{(4)} e_4^{(4)} q_4^{(4)} \]
\[ \vdots \]

The magnitude of the top few values in the first three \( e \)-columns varies between \( 10^{-2} \) and 10 while it drops to machine precision in the fourth \( e \)-column. By (d), this is a clear indication that \( t = 4 \). The value of each term can be obtained by solving a polynomial formed according to (c).

Based on (a) and (b), we recover the non-zero terms in \( p(x, y, z) \) directly from the first four \( q \)-columns. As for the \( e \)-columns, the first three drops from \( 10^{-2} \) to machine precision, while all values in the fourth \( e \)-column are of the order of machine precision. We conclude that \( t = 4 \) by (d).