Sparse Multivariate Polynomial Interpolation via the Quotient-Difference Algorithm

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The qd-scheme

For a given sequence $\{a_n\}$, follow the qd-scheme:

Initialize:

$$e_0^{(n)} = 0, \qquad n = 1, 2, \ldots;$$
$$q_1^{(n)} = \frac{a_{n+1}}{a_n}, \qquad n = 0, 1, 2, \ldots.$$

Continue with:

$$e_m^{(n)} = q_m^{(n+1)} - q_m^{(n)} + e_{m-1}^{(n+1)},$$
$$m = 1, 2, \ldots, \qquad n = 0, 1, 2, \ldots;$$
$$q_{m+1}^{(n)} = \frac{e_m^{(n+1)}}{e_m^{(n)}} q_m^{(n+1)}, \qquad n = 0, 1, 2, \ldots.$$

A two-dimensional array can be formed:

$$\begin{array}{ccccccccc}
 & q_1^{(0)} & & & & & & & \\
0 & & e_1^{(0)} & & & & & & \\
 & q_1^{(1)} & & q_2^{(0)} & & & & & \\
0 & & e_1^{(1)} & & e_2^{(0)} & & & & \\
 & q_1^{(2)} & & q_2^{(1)} & & q_3^{(0)} & & & \\
0 & & e_1^{(2)} & & e_2^{(1)} & & e_3^{(0)} & & \\
\vdots & q_1^{(3)} & \vdots & q_2^{(2)} & \vdots & q_3^{(1)} & \vdots & \ddots & \\
 & \vdots & & \vdots & & \vdots & & &
\end{array}$$

A more stable progressive form exists for the same qd-scheme [3]. In addition, a breakdown free version of the qd-algorithm is described in [1].

References

[1] H. Allouche, A. Cuyt, Reliable pole detection using a deflated qd-algorithm: when Bernoulli, Hadamard and Rutishauser cooperate, Numerische Mathematik, to appear.

[2] A. Cuyt, W.-s. Lee, A new algorithm for sparse interpolation of multivariate polynomials, Theoretical Computer Science, to appear.

[3] P. Henrici, Applied and computational complex analysis I, John Wiley, New York, 1974.

Rutishauser's qd-algorithm

Let the function f be analytic at $z = 0$ and meromorphic in the disk $D : |z| < \delta$. That is, f is analytic on all D except a set of isolated points (poles). Let its poles $z_i = u_i^{-1}$ in D, and let u_i be numbered such that

$$|u_1| \geq |u_2| \geq \cdots \geq |u_t| > 0 = |u_{t+1}|,$$

each pole occurring as many times in the sequence $\{u_k\}$ (or $\{z_k\}$) as indicated by its order.
Let $\{a_n\}$ be the sequence formed by the Taylor coefficients of f

$$f = a_0 + a_1 z + a_2 z^2 + a_3 z^3 \cdots.$$

Then the qd-scheme has the following properties:

(a) for each $0 < j \leq t$ and $|u_j| > |u_{j+1}|$, it holds that
$$\lim_{n\to\infty} e_j^{(n)} = 0;$$

(b) for each $0 < j \leq t$ and $|u_{j-1}| > |u_j| > |u_{j+1}|$, it holds that
$$\lim_{n\to\infty} q_j^{(n)} = u_j = \frac{1}{z_j};$$

(c) for each j and $\ell > 1$ such that $0 < j < j+\ell \leq t$ and $|u_{j-1}| > |u_j| = \cdots = |u_{j+\ell-1}| > |u_{j+\ell}|$, it holds that for the polynomials $\rho_s^{(n)}$ defined by

$$\rho_0^{(n)}(u) = 1,$$
$$\rho_{s+1}^{(n)}(u) = u\rho_s^{(n)}(u) - q_{j+s+1}^{(n)} \rho_s^{(n)}(u),$$
$$n \geq 0, \qquad s = 0, 1, \ldots, \ell - 1,$$

there exists a subsequence that converges to

$$(u - u_j) \cdots (u - u_{j+\ell-1});$$

(d) for $j = t$ we have

$$e_t^{(n)} = 0, \qquad n > 0.$$

Sparse polynomial interpolation via the qd-algorithm

We consider the black box interpolation of polynomial [2]

$$p(x, y, z) = \pi x^5 y^7 z - e y z^{11} - \frac{\sqrt{2}}{10} x^9 z^3 + 100 z^3.$$

Compare to Rutishauser's qd-algorithm for equimodular poles: (c) and (d)
Suppose the number (or an upper bound) of non-zero terms t is unknown. If the partial degree (or its upper bound) in each variable is given, choose $\omega_1 = \exp(2\pi i/17), \omega_2 = \exp(2\pi i/11), \omega_3 = \exp(2\pi i/13)$ and follow the qd-scheme for sequence $\{a_n = p(\omega_1^n, \omega_2^n, \omega_3^n)\}$.

$$\begin{array}{cccccccccc}
 & q_1^{(0)} & & & & & & & & \\
0 & & e_1^{(0)} & & & & & & & \\
 & q_1^{(1)} & & q_2^{(0)} & & & & & & \\
0 & & e_1^{(1)} & & e_2^{(0)} & & & & & \\
 & q_1^{(2)} & & q_2^{(1)} & & q_3^{(0)} & & & & \\
0 & & e_1^{(2)} & & e_2^{(1)} & & e_3^{(0)} & & & \\
 & q_1^{(3)} & & q_2^{(2)} & & q_3^{(1)} & & q_4^{(0)} & & \\
0 & & e_1^{(3)} & & e_2^{(2)} & & e_3^{(1)} & & \mathbf{0} & \\
 & q_1^{(4)} & & q_2^{(3)} & & q_3^{(2)} & & q_4^{(1)} & & \\
0 & & \vdots & & \vdots & & \vdots & & \mathbf{0} &
\end{array}$$

The magnitude of the top few values in the first three e-columns varies between 10^{-2} and 10 while it drops to machine precision in the fourth e-column. By (d), this is a clear indication that $t = 4$. The value of each term can be obtained by solving a polynomial formed according to (c).

Compare to Rutishauser's qd-algorithm for non-equimodular poles: (a), (b), and (d)
When neither t nor (an upper bound of) the partial degree is known, choose $\xi_1 = 1/3, \xi_2 = 1/5, \xi_3 = 1/2$, in which $3, 5, 2$ are pairwise relatively prime, and follow the qd-scheme for sequence $a_n = \{p(\xi_1^n, \xi_2^n, \xi_3^n)\}$.

$$\begin{array}{cccccccccc}
 & q_1^{(0)} & & & & & & & & \\
0 & & e_1^{(0)} & & & & & & & \\
 & q_1^{(1)} & & q_2^{(0)} & & & & & & \\
0 & & e_1^{(1)} & & e_2^{(0)} & & & & & \\
 & q_1^{(2)} & & q_2^{(1)} & & q_3^{(0)} & & & & \\
0 & & e_1^{(2)} & & e_2^{(1)} & & e_3^{(0)} & & & \\
 & q_1^{(3)} & & q_2^{(2)} & & q_3^{(1)} & & q_4^{(0)} & & \\
0 & & e_1^{(3)} & & e_2^{(2)} & & e_3^{(1)} & & \mathbf{0} & \\
 & q_1^{(4)} & & q_2^{(3)} & & q_3^{(2)} & & q_4^{(1)} & & \\
0 & & \downarrow & & \downarrow & & \downarrow & & \vdots & \\
 & \downarrow & 0 & \downarrow & 0 & \downarrow & 0 & \downarrow & \mathbf{0} & \\
 & 1/(2^3) & & 1/(2^{11}5^1) & & 1/(3^9 2^3) & & 1/(3^5 5^7 2^1) & & \\
 & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
 & z^3 & & yz^{11} & & x^9 z^3 & & x^5 y^7 z^1 & &
\end{array}$$

Based on (a) and (b), we recover the non-zero terms in $p(x, y, z)$ directly from the first four q-columns. As for the e-columns, the first three drops from 10^{-2} to machine precision, while all values in the fourth e-column are of the order of machine precision. We conclude that $t = 4$ by (d).