An Iterative Approach toward the Approximate Factorization of Multivariate Polynomials

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1. Univariate case

Suppose $f(x)$ is a univariate polynomial

$$f(x) = (1-\beta_1 x)(1-\beta_2 x)\cdots(1-\beta_t x) \in \mathbb{C}[x]$$

with

$$|\beta_1| > |\beta_2| > \cdots > |\beta_t|.$$

Consider the Taylor expansion of its reciprocal

$$\begin{aligned}\frac{1}{f(x)} &= \frac{r_1}{1-\beta_1 x} + \frac{r_2}{1-\beta_2 x} + \cdots + \frac{r_t}{1-\beta_t x}\\ &= r_1(1+\beta_1 x + \beta_1^2 x + \cdots) + \cdots\\ &\quad + r_t(1+\beta_t x + \beta_t^2 x + \cdots)\\ &= (r_1 + r_2 + \cdots + r_t)\\ &\quad + (r_1\beta_1 + r_2\beta_2 + \cdots + r_t\beta_t)x + \cdots\\ &= \sum_{i=0}^{\infty} a_i x^i,\end{aligned}$$

in which $r_1, \ldots, r_t \in \mathbb{C}$. The i-th Taylor coefficient

$$a_i = r_1\beta_1^i + r_2\beta_2^i + \cdots + r_t\beta_t^i.$$

The sequence $\{a_i\}_{i\geq 0}$ formed by the Taylor coefficients is linear generated by

$$\Lambda(z) = (z-\beta_1)(z-\beta_2)\cdots(z-\beta_t).$$

Since $|\beta_1| > \cdots > |\beta_t|$, as $i \to \infty$, the exponential term $r_1\beta_1^i$ dominates a_i

$$\lim_{i\to\infty} \frac{a_i}{r_1\beta_1^i} = 1.$$

This property can be used to recover all the zeros of $f(x)$ in an order that reflects their moduli [2, pp. 617–618]. Each zero of $f(x)$ corresponds to a linear factor of $f(x)$. A complete factorization of $f(x) \in \mathbb{C}[x]$ can be obtained accordingly.

The general case of $|\beta_1| \geq \cdots \geq |\beta_t|$, in which $|\beta_i| = \cdots = |\beta_j|$ for $1 \leq i \leq j \leq t$, is treated in [2, §7.9].

2. Single multivariate factor

Let f be an irreducible multivariate polynomial with constant normalized to 1. That is,

$$f(x_1, \ldots, x_n) = 1 - p(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$$

with p having zero constant. Expand $1/f$,

$$\begin{aligned}\frac{1}{f} &= \frac{1}{1-p} = 1 + p + p^2 + \cdots\\ &= \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n_{\geq 0}} a_{i_1,\ldots,i_n} x_1^{i_1}\cdots x_n^{i_n} = \sum_{\vec{\imath}\in\mathbb{Z}^n_{\geq 0}} a_{\vec{\imath}} x^{\vec{\imath}}.\end{aligned}$$

Let $p = b_1 x^{\vec{d}_1} + b_2 x^{\vec{d}_2} + \cdots + b_m x^{\vec{d}_m}$. Expand p in the linear recurrence $p^{i+1} = p^i \cdot p$,

$$p^{i+1} = p^i b_1 x^{\vec{d}_1} + \cdots + p^i b_m x^{\vec{d}_m}$$

for $i = M, M+1, \ldots, N$

$$\begin{aligned}p^N &= p^{N-1} b_1 x^{\vec{d}_1} + \cdots + p^{N-1} b_m x^{\vec{d}_m}\\ \vdots &= \qquad\qquad \vdots\\ p^{M+1} &= p^M b_1 x^{\vec{d}_1} + \cdots + p^M b_m x^{\vec{d}_m}\\ p^M &= p^{M-1} b_1 x^{\vec{d}_1} + \cdots + p^{M-1} b_m x^{\vec{d}_m}.\end{aligned}$$

Collect $a_{\vec{k}} x^{\vec{k}}$ from both sides,

$$\begin{aligned}a_{\vec{k}} x^{\vec{k}} &= a_{\vec{k}-\vec{d}_1} x^{\vec{k}-\vec{d}_1} \cdot b_1 x^{\vec{d}_1} + a_{\vec{k}-\vec{d}_2} x^{\vec{k}-\vec{d}_2} \cdot b_2 x^{\vec{d}_2}\\ &\quad + \cdots + a_{\vec{k}-\vec{d}_m} x^{\vec{k}-\vec{d}_m} \cdot b_m x^{\vec{d}_m}\end{aligned}$$

result in

$$a_{\vec{k}} = b_1 a_{\vec{k}-\vec{d}_1} + b_2 a_{\vec{k}-\vec{d}_2} + \cdots + b_m a_{\vec{k}-\vec{d}_m}.$$

Repeat for m various $a_{\vec{k}_j}, \ldots, a_{\vec{k}_{j+m-1}}$

$$\begin{bmatrix} a_{\vec{k}_j-\vec{d}_1} & \cdots & a_{\vec{k}_j-\vec{d}_m}\\ \vdots & \ddots & \vdots\\ a_{\vec{k}_{j+m-1}-\vec{d}_1} & \cdots & a_{\vec{k}_{j+m-1}-\vec{d}_m}\end{bmatrix}\begin{bmatrix} b_1\\ \vdots\\ b_m\end{bmatrix} = \begin{bmatrix} a_{\vec{k}_j}\\ \vdots\\ a_{\vec{k}_{j+m-1}}\end{bmatrix}. \quad (1)$$

3. Dominating factor

Let $f = (1-2x_1-3x_2)(1-x_1-x_2)$, then $1-p_1 = 1-2x_1-3x_2$ is the dominating factor,

$$\begin{aligned}\frac{1}{f} &= \frac{\overbrace{1-2x_1}^{r_1}}{1-\underbrace{(2x_1+3x_2)}_{p_1}} + \frac{\overbrace{-\frac{1}{3}+\frac{2x_1}{3}}^{r_2}}{1-\underbrace{(x_1+x_2)}_{p_2}}\\ &\quad + \frac{\overbrace{\frac{1}{3}+\frac{5x_1}{3}-\frac{2x_1^2}{3}}^{r}}{\underbrace{(1-2x_1-3x_2)}_{1-p_1}\underbrace{(1-x_1-x_2)}_{1-p_2}}\\ &= r_1\left(1+p_1+p_1^2+\cdots\right) + r_2\left(1+p_2+p_2^2+\cdots\right)\\ &\quad + r\left(1+p_1+p_1^2+\cdots\right)\left(1+p_2+p_2^2+\cdots\right)\\ &= \sum_{\vec{\imath}\in\mathbb{Z}^2_{\geq 0}} a_{\vec{\imath}} x^{\vec{\imath}} \quad \text{with } r_1, r_2, r \in \mathbb{C}[x_1, x_2].\end{aligned}$$

The contribution due to the powers of p_1 dominates $a_{\vec{\imath}}$. For a sufficiently large $j > 0$, coefficients $b_1, \ldots, b_m$ in $1-p_1$ are approximated by solving (1).

5. Non-dominating factors

Distinct non-dominating factors.
For example, $f = (1-x_1-x_2)(1-x_1+x_2)$.

$\Longrightarrow$ **partially dominating factors:**
Let $y_1 = -3+x_1+x_2, y_2 = 18-2x_1-4x_2$. Then $f = (-2-y_1)(10-3y_1-y_2)$ and after normalization $(1+0.5y_1)(1-0.3y_1-0.1y_2)$.

$\Longrightarrow$ **dominating factor:**
Let $z_1 = -3+x_1+x_2, z_2 = 10-2x_1-4x_2$. Then $f = (-2-z_1)(2-3z_1-z_2)$ and after normalization $(1+0.5z_1)(1-1.5z_2-0.5z_2)$.

Distinct non-dominating factors.
Let $f = (1-p_1)^s \in \mathbb{C}[x_1, \ldots, x_n]$. Form $g = (1-\bar{p}_1)^s \in \mathbb{C}[y_1, \ldots, y_n]$ by using $y_1, \ldots, y_n$ to replace $x_1, \ldots, x_n$ in f. Define $F = f - g \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. If $s = 1$, then F is irreducible.

If $s \geq 2$, then $F = f - g = (1-p_1)^s - (1-\bar{p}_1)^s$

$$= (\bar{p}_1 - p_1)\sum_{j=0}^{s-1}(1-p_1)^j(1-\bar{p}_1)^{s-1-j}.$$

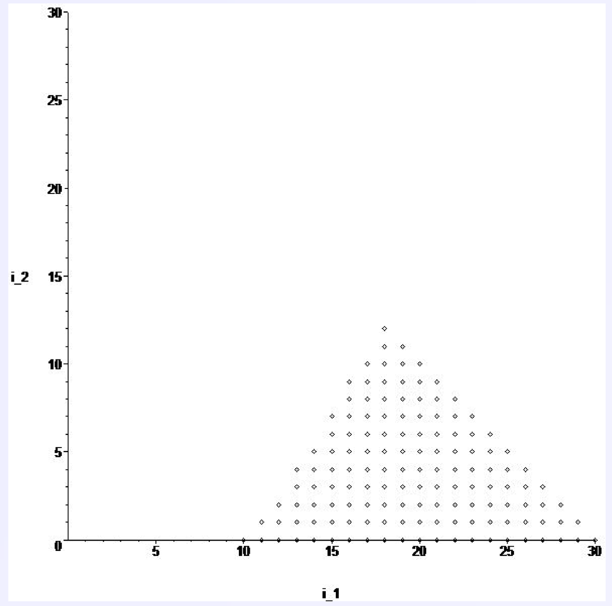
4. Partially dominating factors

Let $f = (1-x_1+x_2)(1-7x_1)$. Both $1-p_1$ and $1-p_2$ partially dominate,

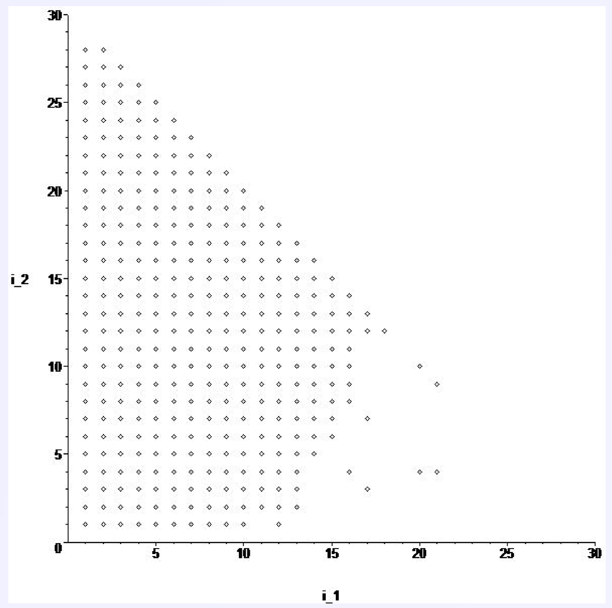
$$\begin{aligned}\frac{1}{f} &= \frac{r_1}{1-\underbrace{(x_1-x_2)}_{p_1}} + \frac{r_2}{1-\underbrace{7x_1}_{p_2}}\\ &\quad + \frac{r}{\underbrace{(1-x_1+x_2)(1-7x_1)}_{(1-p_1)(1-p_2)}} = \sum_{\vec{\imath}\in\mathbb{Z}^2_{\geq 0}} a_{\vec{\imath}} x^{\vec{\imath}}.\end{aligned}$$

The contribution due to either the powers of p_1 or p_2 dominates in a respective subset of the Taylor coefficients a_{i_1,i_2}.

Let the computed factor be $1-\phi$. The computational environment is Maple 12 with Digits $:= 15$.



The dominating behavior of $1-p_2$ is recorded by dotted (i_1, i_2) where $\|\phi - p_2\|_2 < 10^{-7}$ and $i_1 + i_2 = 0, \ldots, 30$.



The dominating behavior of $1-p_1$ is recorded by dotted (i_1, i_2) where $\|\phi - p_1\|_2 < 10^{-7}$ and $i_1 + i_2 = 0, \ldots, 30$.

6. References

[1] Cuyt, A., and Lee, W.-s. Extracting approximate factors of multivariate polynomials from Taylor expansions. Submitted.

[2] Henrici, P. Applied and computational complex analysis I. John Wiley, New York, 1974.