# From Ben-Or/Tiwari to QD and Generalized Eigenvalues I

Annie Cuyt    Wen-shin Lee    Brigitte Verdonk
Department of Mathematics and Computer Science, University of Antwerp, Belgium

## Ben-Or/Tiwari algorithm $f(x,y) = 3x^3y^2 + 15y - 7x + 5$

- Pick $p_1, p_2$ relatively prime: say $p_1 = 2,\ p_2 = 3$.
  Evaluate $a_0 = f(2^0, 3^0) = 252, a_1 = f(2,3) = 15664, a_2 = f(2^2, 3^2) = 1120098, a_3 = f(2^3, 3^3) = 80622676, \ldots$.

  sequence $a_0, a_1, a_2, \ldots$ is linearly generated

- Compute the minimal generator $\Lambda(z)$ by Berlekamp/Massey algorithm or solving a Hankel system.

  $$\Lambda(z) = z^4 - 78z^3 + 443z^2 - 798z + 432 = (z-1)(z-2)(z-3)(z-72)$$

- Roots of $\Lambda(z) = 0 : 1, 2, 3, 72$ are non-zero terms in $f$ at $x = 2, y = 3$:

  $$1 = 2^0 \cdot 3^0 \Rightarrow 1, \quad 2 = 2 \cdot 3^0 \Rightarrow x, \quad 3 = 2^0 \cdot 3 \Rightarrow y, \quad 72 = 2^3 \cdot 3^2 \Rightarrow x^3y^2$$

- Recover coefficients by solving a Vandermonde system.

## Generalized eigenvalue problem

$$\underbrace{\begin{bmatrix} a_{j+1} & a_{j+2} & a_{j+3} & a_{j+4} \\ a_{j+2} & a_{j+3} & a_{j+4} & a_{j+5} \\ a_{j+3} & a_{j+4} & a_{j+5} & a_{j+6} \\ a_{j+4} & a_{j+5} & a_{j+6} & a_{j+7} \end{bmatrix}}_{H_4^{(j+1)}} v = \beta \underbrace{\begin{bmatrix} a_j & a_{j+1} & a_{j+2} & a_{j+3} \\ a_{j+1} & a_{j+2} & a_{j+3} & a_{j+4} \\ a_{j+2} & a_{j+3} & a_{j+4} & a_{j+5} \\ a_{j+3} & a_{j+4} & a_{j+5} & a_{j+6} \end{bmatrix}}_{H_4^{(j)}} v$$

Generalized eigenvalue solutions for $\beta$ are $72, 3, 2, 1$.

## Generating Polynomial (Hadamard polynomial)

$$\begin{aligned} \Lambda(z) &= z^4 - 78z^3 + 443z^2 - 798z + 432 \\ &= (z-1)(z-2)(z-3)(z-72) \end{aligned}$$

$$\Longrightarrow a_{j+4} = 78a_{j+3} - 443a_{j+2} + 798a_{j+1} - 432a_j$$

## The QD algorithm

$$g(z) = \frac{-2(498z - 8 - 1548z^2 + 1413z^3)}{(1-72z)(1-3z)(1-2z)(1-z)} = \frac{3}{1-72z} + \frac{15}{1-3z} - \frac{7}{1-2z} + \frac{5}{1-z} = a_0 + a_1z + a_2z^2 + a_3z^3 + \cdots$$

$$a_j = 3 \cdot (72)^j + 15 \cdot (3)^j - 7 \cdot (2)^j + 5 \cdot (1)^j = 3 \cdot (2^j)^3 \cdot (3^j)^2 + 15 \cdot (3^j) - 7(2^j) + 5 \cdot (1^j) = f(2^j, 3^j)$$

Function $g(z)$ has poles at $\frac{1}{72}, \frac{1}{3}, \frac{1}{2}, 1$.

- Initialize:

$$e_0^{(n)} = 0, \quad n = 1, 2, \ldots; \qquad q_1^{(n)} = \frac{a_{n+1}}{a_n}, \quad n = 0, 1, 2, \ldots.$$

- Continue with:

$$e_m^{(n)} = q_m^{(n+1)} - q_m^{(n)} + e_{m-1}^{(n+1)}, \quad m = 1, 2, \ldots, \quad n = 0, 1, 2, \ldots; \qquad q_{m+1}^{(n)} = \frac{e_m^{(n+1)}}{e_m^{(n)}} q_m^{(n+1)}, \quad n = 0, 1, 2, \ldots.$$

$$\begin{array}{cccccccc}
 & q_1^{(0)} & & & & & & \\
0 & & e_1^{(0)} & & & & & \\
 & q_1^{(1)} & & q_2^{(0)} & & & & \\
0 & & e_1^{(1)} & & e_2^{(0)} & & & \\
 & q_1^{(2)} & & q_2^{(1)} & & q_3^{(0)} & & \\
0 & & e_1^{(2)} & & e_2^{(1)} & & e_3^{(0)} & \\
 & q_1^{(3)} & & q_2^{(2)} & & q_3^{(1)} & & \ddots \\
0 & & \downarrow & & \downarrow & & \downarrow & \\
 & \downarrow & 0 & \downarrow & 0 & \downarrow & 0 & \downarrow \\
 & 72 & & 3 & & 2 & & 1
\end{array}$$