From Quotient-Difference to Generalized Eigenvalues and Sparse Polynomial Interpolation

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ABSTRACT

The numerical quotient-difference algorithm, or the qd-algorithm, can be used for determining the poles of a meromorphic function directly from its Taylor coefficients.

We show that the poles computed in the qd-algorithm, regardless of their multiplicities, are converging to the solution of a generalized eigenvalue problem. In a special case when all the poles are simple, such generalized eigenvalue problem can be viewed as a reformulation of Prony's method, a method that is closely related to the Ben-Or/Tiwari algorithm for interpolating a multivariate sparse polynomial in computer algebra.

Categories and Subject Descriptors

G.1.2 [**NUMERICAL ANALYSIS**]: Approximation—*Rational approximation*; I.1.2 [**SYMBOLIC AND ALGEBRAIC MANIPULATION**]: Algorithms—*Algebraic algorithms*

General Terms

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Keywords

Ben-Or/Tiwari algorithm, Prony's method, generalized eigenvalue, qd-algorithm, sparse polynomial interpolation

1. INTRODUCTION

In simulation and modeling problems, an analytic model is often used to approximate a complex system. A meromorphic function is a function that is analytic everywhere except at a set of isolated points, which are poles of the function. In many applications, detecting poles is critical, in which variances of the quotient-difference algorithm developed by Rutishauser [20] (or see, e.g., [11, 13]) have been widely implemented.

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The quotient-difference algorithm, or the qd-algorithm, is an iterative numerical scheme for determining the poles of a meromorphic function $f(z)$ from its Taylor series, namely,

$$f(z) = \sum_{i=0}^{\infty} \tilde{a}_i z^i.$$

Many achievements have been built upon the original qd-algorithm. Among them, the qd-algorithm can be regarded as computing the eigenvalues for a tridiagonal matrix. This is based on a matrix interpretation of the formally orthogonal Hadamard polynomials associated with Taylor coefficients [11, pp. 634–636]. Several further improvements can be found in [8, 18, 21, 22].

Another approach by Kravanja and Van Barel [15, 16] separately considers mutually distinct poles and their respective multiplicities from the evaluation of numerical integration. These distinct poles are computed as simple poles by solving a generalized eigenvalue problem. Their respective multiplicities are determined via solving a Vandermonde system.

We give an additional interpretation to the qd-algorithm, in which the poles, regardless of their multiplicities, are directly viewed as solutions to a generalized eigenvalue problem of two shifted Hankel systems. When all the poles are simple, such problem of generalized eigenvalue is comparable to a reformulation of Prony's method, a method that can interpolate a sum of exponential functions [10, 19] and closely related to the Ben-Or/Tiwari algorithm for interpolating a multivariate sparse polynomial in computer algebra [1, 9]. We notice the analogy between the vanishing of e columns in the qd-algorithm and the zero discrepancy in early termination strategy for sparse polynomial interpolation [14].

For later reference, in Section 2 we briefly describe the qd-algorithm, Prony's method, and the Ben-Or/Tiwari algorithm. In Section 3 we show our interpretation of the poles as generalized eigenvalues. In Section 4 we conclude with our current research.

2. THE QD-ALGORITHM AND BEN-OR/TIWARI INTERPOLATION

2.1 The quotient-difference algorithm

If the linear functionals a and $a^{(n)}$ are defined from the vector space $\mathbb{C}[z]$ to $\mathbb{C}$ by $a(z^i) = a_i$, $a^{(n)}(z^i) = a_{i+n}$ for $i, n \geq 0$, it is clear that a and $a^{(n)}$ can be completely determined by the sequence $\{a_i\}_{i\in\mathbb{Z}_{\geq 0}}$.

Consider the following Hankel determinants associated with $\{a_i\}_{i\in\mathbb{Z}_{\geq 0}}$:

$$\det H_t^{(n)} = \det \begin{bmatrix} a_n & \cdots & a_{n+t-1} \\ \vdots & \ddots & \vdots \\ a_{n+t-1} & \cdots & a_{n+2t-2} \end{bmatrix}, \det H_0^{(n)} = 1,$$

$$\det H_t^{(n)}(u) = \det \begin{bmatrix} a_n & a_{n+1} & \cdots & a_{n+t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+t-1} & a_{n+t} & \cdots & a_{n+2t-1} \\ 1 & u & \cdots & u^t \end{bmatrix},$$

$$\det H_0^{(n)}(u) = 1,$$

and the Hadamard polynomials

$$\Lambda_t^{(n)}(u) = \frac{\det H_t^{(n)}(u)}{\det H_t^{(n)}}, \quad t \geq 0,\ n \geq 0.$$

The monic polynomials $\Lambda_t^{(n)}(u)$ are formally orthogonal with respect to the linear functional $a^{(n)}$. That is, $a^{(n)} \cdot u^j \Lambda_t^{(n)}(u) = 0$, for $j = 0, \ldots, t-1$ [2, pp. 40–41]. The functional a is called T-normal if $\det H_t^{(n)} \neq 0$, for all $n \geq 0$ and $t = 0, \ldots, T$. From now on we assume that this is the case.

Let $\{a_i\}_{i\in\mathbb{Z}_{\geq 0}}$ be the sequence from a field associated with a formal power series $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots$. With subscripts denoting columns and superscripts downward sloping diagonals, the qd-scheme initializes

$$e_0^{(n)} = 0 \text{ for } n = 1, 2, \ldots,$$
$$q_1^{(n)} = \frac{a_{n+1}}{a_n} \text{ for } n = 0, 1, \ldots,$$

and continues for $t = 1, 2, \ldots$ and $n = 0, 1, \ldots$ by the rhombus rules

$$e_t^{(n)} = q_t^{(n+1)} - q_t^{(n)} + e_{t-1}^{(n+1)},$$
$$q_{t+1}^{(n)} = \frac{e_t^{(n+1)}}{e_t^{(n)}} q_t^{(n+1)},$$

when the corresponding quotients and differences exist.

According to [11, Theorem 7.6a], if F is T-normal for $T > 0$, the columns $q_t^{(n)}$ associated with F exist, and

$$q_t^{(n)} = \frac{\det H_t^{(n+1)} \cdot \det H_{t-1}^{(n)}}{\det H_t^{(n)} \cdot \det H_{t-1}^{(n+1)}},$$
$$e_t^{(n)} = \frac{\det H_{t+1}^{(n)} \cdot \det H_{t-1}^{(n+1)}}{\det H_t^{(n)} \cdot \det H_t^{(n+1)}}$$

for $t = 1, 2, \ldots, T$ and all $n \geq 0$.

The row-wise generation of the qd-scheme is more stable. It initializes with

$$e_0^{(n)} = 0, \quad n = 1, 2, \ldots,$$
$$q_t^{(0)} = \frac{\det H_{t-1}^{(0)} \cdot \det H_t^{(1)}}{\det H_t^{(0)} \cdot \det H_{t-1}^{(1)}},$$
$$e_t^{(0)} = \frac{\det H_{t+1}^{(0)} \cdot \det H_{t-1}^{(1)}}{\det H_t^{(0)} \cdot \det H_t^{(1)}}, \quad t = 1, 2, \ldots,$$

and continues for $n = 0, 1, 2, \ldots$ with

$$q_t^{(n+1)} = e_t^{(n)} - e_{t-1}^{(n+1)} + q_t^{(n)},$$
$$e_t^{(n+1)} = \frac{q_{t+1}^{(n)}}{q_t^{(n+1)}} e_t^{(n)}.$$

Theorem 1. [11, Theorem 7.6b, the qd-algorithm] *Let $F = \sum_{i=0}^{\infty} a_i z^i$ be the Taylor series at $z = 0$ of a function f meromorphic in the disk $D : |z| < \sigma$ and let the poles $z_j = u_j^{-1}$ of f in D be numbered such that $0 < |z_1| \leq |z_2| \leq \cdots < \sigma$, each pole occurring as many times in the sequence $\{z_j\}$ as indicated by its order. If F is ultimately T-normal for some integer $T > 0$, then the qd-scheme associated with F has the following properties:*

(a) For each t such that $0 < t \leq T$ and $|z_{t-1}| < |z_t| < |z_{t+1}|$ (where $z_0 = 0$ and, if f has only T poles, $z_{T+1} = \infty$),

$$\lim_{n\to\infty} q_t^{(n)} = u_t;$$

(b) For each m such that $0 < t \leq T$ and $|z_t| < |z_{t+1}|$,

$$\lim_{n\to\infty} e_t^{(n)} = 0.$$

2.2 Prony's method and the Ben-Or/Tiwari algorithm

Prony's method [12, 19] interpolates a univariate function $\varrho(x)$ as a sum of exponential functions. It seeks to determine c_j and α_j such that $\varrho(x) = \sum_{j=1}^{t} c_j e^{\alpha_j x}$ with $c_j \neq 0$ from evaluations of $\varrho(x)$ at $2t$ equally spaced points, for example, $\varrho(0)$, $\varrho(1)$ ... $\varrho(2t-1)$.

Let $b_j = e^{\alpha_j}$ and consider the monic polynomial $\Lambda_t(u)$ having b_j's as zeros: $\Lambda_t(u) = \prod_{j=1}^{t}(u - b_j) = u^t + \lambda_{t-1}u^{t-1} + \cdots \lambda_1 u + \lambda_0$. A $t \times t$ Hankel system can be obtained by summing up $\lambda_0 \cdot \varrho(j), \lambda_1 \cdot \varrho(j+1), \ldots, \lambda_{t-1} \cdot \varrho(j+t-1), \varrho(j+t)$ for $j \geq 0$. The solutions to this Hankel system are coefficients of the polynomial $\Lambda_t(u)$:

$$\begin{bmatrix} \varrho(0) & \ldots & \varrho(t-1) \\ \vdots & \ddots & \vdots \\ \varrho(t-1) & \ldots & \varrho(2t-2) \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \vdots \\ \lambda_{t-1} \end{bmatrix} = - \begin{bmatrix} \varrho(t) \\ \vdots \\ \varrho(2t-1) \end{bmatrix}. \quad (1)$$

To determine $\Lambda_t(u)$, Prony's method solves (1). Then $b_1, \ldots, b_t$ (hence $\alpha_1, \ldots, \alpha_t$) can be obtained by the root finding on this polynomial. The coefficients $c_1, \ldots, c_t$ in $\varrho(x)$ are computed by solving a transposed Vandermonde system:

$$\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ b_1^{t-1} & \cdots & b_t^{t-1} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_t \end{bmatrix} = \begin{bmatrix} \varrho(0) \\ \vdots \\ \varrho(t-1) \end{bmatrix}.$$

As for the problem of sparse polynomial interpolation, consider a black box polynomial p with n variables, the Ben-Or/Tiwari method [1] finds coefficients c_j and integer exponents $(d_{j_1}, \ldots, d_{j_n})$ such that

$$p(x_1, \ldots, x_n) = \sum_{j=1}^{t} c_j x_1^{d_{j_1}} \cdots x_n^{d_{j_n}} \text{ with } c_j \neq 0.$$

Let $\beta_j(x_1, \ldots, x_n) = x_1^{d_{j_1}} \cdots x_n^{d_{j_n}}$ be the j-th term in $p(x_1, \ldots, x_n)$ and $b_j = \beta_j(\omega_1, \ldots, \omega_n)$ with $\omega_1, \ldots, \omega_n$ pairwise relatively prime, then $b_j^i = \beta_j(\omega_1^i, \ldots, \omega_n^i)$ for any power $i \geq$

0. The Ben-Or/Tiwari algorithm considers $\varrho(i) = p(\omega_1^i, \ldots, \omega_n^i)$ similarly to the Prony's method: it finds a generating polynomial $\Lambda_t(u)$, determines b_j by root finding, and solves an associated transposed Vandermonde system for c_j.

Once the nonzero terms b_j are found through the root finding of $\Lambda_t(u) = 0$, the multivariate exponents $(d_{j_1}, \ldots, d_{j_n})$ can be determined via repeatedly dividing b_j by $\omega_1, \ldots, \omega_n$ that are relatively prime. This step is based on the exact arithmetic; the Ben-Or/Tiwari algorithm in floating-point arithmetic is developed in [9].

3. THE POLES AS GENERALIZED EIGENVALUES

We show that the poles computed in the qd-algorithm converge to the solution to a generalized eigenvalue problem. We first present the case when all the poles are simple, then extend our result to a general case in which poles may have order larger than 1.

3.1 All the poles are simple

Consider the case when all the poles of a meromorphic function $f(z)$ are simple,

$$f(z) = \underbrace{\frac{r_1}{z_1 - z} + \frac{r_2}{z_2 - z} + \cdots + \frac{r_t}{z_t - z}}_{\phi(z)} + g(z), \qquad (2)$$

where $0 < |z_1| < |z_2| < \cdots < |z_t| < \delta$ and g is analytic in $|z| < \delta$. Let $u_j = z_j^{-1}$, the coefficients in the Taylor expansion for $\phi(z) = \sum_{i=0}^{\infty} a_i z^i$ are:

$$\begin{aligned} a_i &= r_1 z_1^{-i-1} + r_2 z_2^{-i-1} + \cdots + r_t z_t^{-i-1} \\ &= r_1 u_1^{i+1} + r_2 u_2^{i+1} + \cdots + r_t u_t^{i+1}. \end{aligned} \qquad (3)$$

Consider the Hankel matrices $H_t^{(n)}$ associated with $\{a_i\}_{i \in \mathbb{Z}_{\geq 0}}$:

$$\begin{aligned} H_t^{(n)} &= \begin{bmatrix} a_n & \cdots & a_{n+t-1} \\ \vdots & \ddots & \vdots \\ a_{n+t-1} & \cdots & a_{n+2t-2} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=1}^{t} r_j u_j^{n+1} & \cdots & \sum_{j=1}^{t} r_j u_j^{n+t} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{t} r_j u_j^{n+t} & \cdots & \sum_{j=1}^{t} r_j u_j^{n+2t-1} \end{bmatrix}. \end{aligned} \qquad (4)$$

The matrix $H_t^{(n)}$ can be factorized as $H_t^{(n)} = \mathcal{V}\mathcal{D}\overline{\mathcal{V}}^{\mathrm{Tr}}$, where

$$\mathcal{V} = \begin{bmatrix} 1 & \cdots & 1 \\ u_1 & \cdots & u_t \\ \vdots & \ddots & \vdots \\ u_1^{t-1} & \cdots & u_t^{t-1} \end{bmatrix}, \mathcal{D} = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & r_t \end{bmatrix},$$

$$\overline{\mathcal{V}}^{\mathrm{Tr}} = \begin{bmatrix} u_1^{n+1} & \cdots & u_1^{n+t} \\ u_2^{n+1} & \cdots & u_2^{n+t} \\ \vdots & \ddots & \vdots \\ u_t^{n+1} & \cdots & u_t^{n+t} \end{bmatrix}.$$

Combining with the factorization of $H_t^{(n+1)} = \mathcal{V}\mathcal{D}\mathcal{Z}\overline{\mathcal{V}}^{\mathrm{Tr}}$, in which

$$\mathcal{Z} = \begin{bmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_t \end{bmatrix},$$

we can obtain $u_j = 1/z_j$ by solving a generalized eigenvalue problem: $H_t^{(n+1)} v = u H_t^{(n)} v$. In other words,

$$H_t^{(n)} v = z H_t^{(n+1)} v \qquad (5)$$

has solutions $z_1, \ldots, z_t$ that are the t simple poles of $\phi(z)$.

THEOREM 2. *Let function $f(z)$ be analytic at $z = 0$ and meromorphic in the disk $D : |z| < \delta$. Suppose its poles $z_j = u_j^{-1}$ in D are all simple and numbered as $0 < |z_1| < |z_2| < \cdots < |z_t| < \delta < |z_{t+1}|$. (If f has only t poles, then $z_{t+1} = \infty$.) Consider the Taylor expansion of $f(z) = \sum_{i=0}^{\infty} \tilde{a}_i z^i$ and denote the Hankel matrix associated with $\{\tilde{a}_i\}_{i \in \mathbb{Z}_{\geq 0}}$ as $\mathcal{H}_t^{(n)}$, then the t solutions of the generalized eigenvalue problem*

$$\mathcal{H}_t^{(n)} v = z \mathcal{H}_t^{(n+1)} v \qquad (6)$$

converge to the first t poles of $f(z)$: $z_1, \ldots, z_t$ as $n \to \infty$.

PROOF. Consider $f(z) = \phi(z) + g(z)$ in (2). Let $g(z) = \sum_{i=0}^{\infty} b_i z^i$, then $f(z) = \sum_{i=0}^{\infty} \tilde{a}_i z^i = \sum_{i=0}^{\infty} (a_i + b_i) z^i$ for a_i in (3). By the Cauchy coefficient estimate, there is a value $\eta > 0$ such that $|b_n| \leq \eta \rho^n$ for $|1/z_t| = |u_t| > \rho > 1/\delta$ and for all $n \geq 0$. Since $1 = |u_1/u_1| > \cdots > |u_t/u_1| > |\rho/u_1| > 0$, for any $\epsilon > 0$ there exit $\zeta \in \mathbb{Z}_{\geq 0}$ such that for all $n > \zeta$

$$\begin{aligned} \frac{|b_n|}{|a_n|} &\leq \left| \frac{\eta \rho^n}{|r_1 u_1^{n+1}| - |\sum_{j=2}^{t} r_j u_j^{n+1}|} \right| \\ &= \left| \frac{\eta (\rho/u_1)^n}{|r_1 u_1| - |\sum_{j=2}^{t} r_j u_j (u_j/u_1)^n|} \right| < \epsilon. \end{aligned} \qquad (7)$$

If we denote the $t \times t$ Hankel matrix associated with $\{b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ as $B_t^{(n)}$ and define $H_t^{(n)}$ as in (4), then for all $n > \zeta$, $B_t^{(n)}$ and $B_t^{(n+1)}$ can be regarded as the ϵ-perturbations of the generalized eigenvalue problem in (5)

$$\begin{aligned} &\overbrace{(H_t^{(n)} + B_t^{(n)})}^{\mathcal{H}_t^{(n)}}(v + \epsilon v^{(1)} + \cdots) \\ &= (z + \epsilon z^{(1)} + \cdots)\underbrace{(H_t^{(n+1)} + B_t^{(n+1)})}_{\mathcal{H}_t^{(n+1)}}(v + \epsilon v^{(1)} + \cdots). \end{aligned}$$

Then we can follow the discussion of (3.20) in [10]. □

In the disk of $D : |z| < \delta$, we have inequality (7) that provides the convergence to the generalized eigenvalue problem $H_t^{(n)} v = z H_t^{(n+1)} v$. Such inequality gives the convergence to another linear system $H_t^{(n)} \lambda = [a_{n+t}, \ldots, a_{n+2t-1}]^{\mathrm{Tr}}$ that is linked to Prony's method [19] and the Ben-Or/Tiwari algorithm for sparse polynomial interpolation [1].

THEOREM 3. *Let function $f(z)$ be analytic at $z = 0$, meromorphic in the disk $D : |z| < \delta$, and its Taylor expansion $f(z) = \sum_{j=0}^{\infty} \tilde{a}_j z^j$. Suppose its poles $z_j = u_j^{-1}$ in D are all simple and numbered as $0 < |z_1| < |z_2| < \cdots < |z_t| < \delta <$*

$|z_{t+1}|$ *(if f has only t poles, then $z_{t+1}=\infty$) and consider the polynomial $\Lambda_t^{(n)}(u) = u^t + \lambda_{t-1}^{(n)} u^{t-1} + \cdots + \lambda_0^{(n)}$ with coefficients formed from the solution of the Hankel system associated with $\{\tilde{a}_i\}_{i\in\mathbb{Z}_{\geq 0}}$*

$$\underbrace{\begin{bmatrix} \tilde{a}_n & \tilde{a}_{n+1} & \dots & \tilde{a}_{n+t-1} \\ \tilde{a}_{n+1} & \tilde{a}_{n+2} & \dots & \tilde{a}_{n+t} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n+t-1} & \tilde{a}_{n+t} & \dots & \tilde{a}_{n+2t-2} \end{bmatrix}}_{\mathcal{H}_t^{(n)}} \begin{bmatrix} \lambda_0^{(n)} \\ \lambda_1^{(n)} \\ \vdots \\ \lambda_{t-1}^{(n)} \end{bmatrix} = - \begin{bmatrix} \tilde{a}_{n+t} \\ \tilde{a}_{n+t+1} \\ \vdots \\ \tilde{a}_{n+2t-1} \end{bmatrix}. \tag{8}$$

Then the polynomial $\Lambda_t^{(n)}(u) = u^t + \lambda_{t-1}^{(n)} u^{t-1} + \cdots + \lambda_0^{(n)}$ converges to the polynomial $\Lambda_t(u) = (u-u_1)\cdots(u-u_t)$, whose zeros are the reciprocals of the first t poles of $f(z)$, as $n \to \infty$.

Proof. Similar to the proof of Theorem 2, for any $\epsilon > 0$ there exists $\zeta \in \mathbb{Z}_{\geq 0}$ such that for all $i > \zeta$, $|b_i|/|a_i| < \epsilon$. Thus, b_i can be regarded as errors incurred in estimating $a_i = \sum_{j=1}^t r_j u_j^{i+1}$, that is $\tilde{a}_i = a_i + b_i$. Then we consider Prony's method for determining the sum of exponential functions: $\varrho(i) = a_i = \sum_{j=1}^t r_j u_j^{i+1}$. $\square$

The polynomial $\Lambda_t^{(n)}(u)$ in Theorem 3 is the associated Hadamard polynomial for the Taylor series $f(z) = \sum_{i=0}^\infty \tilde{a}_i z^i$. As for the Hankel system associated with the rational part $\phi(z) = \sum_{i=0}^\infty a_i z^i$ of $f(z)$ (also see (2) and (3)), we have the following two lemmas. (Note that $\phi(z) = \sum_{i=0}^\infty a_i z^i$ has exactly t poles.)

Lemma 1. *The Hankel matrix $H_i^{(n)}$ associated with sequence $\{a_i\}_{i\in\mathbb{Z}_{\geq 0}}$ is singular for all $i \geq t+1$.*

Proof. When $i \geq t+1$, the $(t+1)$-th column in $H_i^{(n)}$ is a linear combination of the first t columns and is thus singular. $\square$

While $H_i^{(n)}$ is singular for all $i \geq t+1$, matrix $H_i^{(n)}$ is non-singular for $0 \leq i \leq t$.

Lemma 2. *The Hankel system $H_i^{(n)}$ associated with sequence $\{a_i\}_{i\in\mathbb{Z}_{\geq 0}}$ is non-singular for all $0 \leq i \leq t$ and $n > 0$.*

Proof. By definition, $\det H_0^{(n)} = 1$. Based on Theorem 2 in [14], $H_i^{(n)}$ is non-singular for $1 \leq i \leq t$ and all $n > 0$. $\square$

While the singularities in Lemmas 1 and 2 are related to the e columns in the qd-algorithm [6], such singularities are also comparable to discrepancies in the early termination for the Ben-Or/Tiwari sparse interpolation [14]. In the exact arithmetic environment, the early termination is detected via finding the first zero discrepancy in the Berlekamp/Massey algorithm [17]. The vanishing of discrepancies is reflected as a zero determinant of an associated Hankel system, which is certainly related to the singularity of our corresponding Hankel system. The argument is based on an interpretation of the Berlekamp/Massey algorithm as the extended Euclidean algorithm [7] and the fundamental theorem on subresultant [3]. Further details can be found in [14].

3.2 Poles with multiple orders

To deal with the general case in which the meromorphic function $f(z)$ may have poles of order > 1 in the disk $D : |z| < \delta$, we let $f(z)$ have a multiple pole of order $m > 1$, say at z_1, and all other poles $z_{m+1}, \dots, z_t$ simple. We shall develop our results on such $f(z)$, since it is straightforward to further extend the results to meromorphic functions with different combinations of poles with various orders.

Let $f(z) = \phi(z) + g(z)$, where $g(z) = \sum_{i=0}^\infty b_i z^i$ is analytic in $|z| < \delta$, $0 < |z_1| < |z_{m+1}| < \cdots < |z_t| < \delta$, and

$$\begin{aligned}\phi(z) = &\frac{r_1}{z_1 - z} + \frac{r_2}{(z_1-z)^2} + \cdots + \frac{r_m}{(z_1-z)^m} \\ &+ \frac{r_{m+1}}{z_{m+1}-z} + \cdots + \frac{r_t}{z_t - z}.\end{aligned}$$

Let $u_j = z_j^{-1}$ and consider coefficients in $\phi(z) = \sum_{i=0}^\infty a_i z^i$, then

$$\begin{aligned} a_i = &\underbrace{r_1 u_1^{i+1} + \cdots + r_k \binom{i+k-1}{k-1} u_1^{i+k} + \cdots + r_m \binom{i+m-1}{m-1} u_1^{i+m}}_{\sigma_i} \\ &+ r_{m+1} u_{m+1}^{i+1} + \cdots + r_t u_t^{i+1}. \end{aligned} \tag{9}$$

Lemma 3 allows us to express σ_{i+s} in the same binomial basis used in (9).

Lemma 3. *For $s \geq 1$,*

$$\begin{aligned} \sigma_{i+s} = &\left(\sum_{j=1}^m \binom{j+s-2}{s-1} r_j u_1^j\right) u_1^{i+s} + \cdots \\ &+ \left(\sum_{j=k}^m \binom{j+s-k-1}{s-1} r_j u_1^j\right) \binom{i+k-1}{k-1} u_1^{i+s} + \cdots \\ &+ r_m u_1^m \binom{i+m-1}{m-1} u_1^{i+s}. \end{aligned} \tag{10}$$

Proof. Consider the following array:

$$\begin{array}{cccc} r_1 u_1^{i+1} & r_2 \binom{i+1}{1} u_1^{i+2} & \cdots & r_m \binom{i+m-1}{m-1} u_1^{i+m} \\ r_1 u_1^{i+2} & r_2 \binom{i+2}{1} u_1^{i+3} & \cdots & r_m \binom{i+m}{m-1} u_1^{i+m+1} \\ \vdots & \vdots & \ddots & \vdots \\ r_1 u_1^{i+s+1} & r_2 \binom{i+s+1}{1} u_1^{i+s+2} & \cdots & r_m \binom{i+m+s-1}{m-1} u_1^{i+m+s} \\ \vdots & \vdots & \ddots & \vdots \end{array}$$

and let j start with 0, then the sum of entries in the j-th row is σ_{i+j}. We use $\ell = 0, \dots, m-1$ to enumerate entries in each j-th row.

Look at the s-th row, our purpose is to express the binomial part of each ℓ-th entry as binomials from the 0-th row. These binomials also appear in (9). They are $\binom{i+1}{1}$, $\binom{i+2}{2}$, $\dots$, $\binom{i+m-1}{m-1}$.

Due to the rule of sum: $\binom{i+s+\ell}{\ell} = \binom{i+s+\ell-1}{\ell} + \binom{i+s+\ell-1}{\ell-1}$, the binomial in the (s,ℓ)-th entry is the sum of binomials in $(s,\ell-1)$-th and $(s-1,\ell)$-th entries. We note that the binomial part forms a Pascal's triangle.

Repeatedly applying the rule of sum to every entry in the s-th row, eventually we will be able to express the binomial part completely in the binomials from the 0-th row. We obtain (10) by collecting them with respect to similar binomial basis for σ_i in (9): u_1^{i+s}, $\binom{i+1}{1} u_1^{i+s}$, $\binom{i+2}{2} u_1^{i+s}$, $\dots$, $\binom{i+m-1}{m-1} u_1^{i+s}$ and summing them up accordingly. $\square$

Consider another expression for σ_i: for $1 \le k \le m$,

$$\sigma_i = r_1 \cdot u_1^{i+1} + \cdots + \frac{r_k}{(k-1)!} \cdot \Big((i+k-1)\cdots(i+1)u_1^{i+k}\Big) + \cdots + \frac{r_m}{(m-1)!} \cdot \Big((i+m-1)\cdots(i+1)u_1^{i+m}\Big). \tag{11}$$

We can represent σ_{i+s} in Lemma 3 in a similar manner:

$$\sigma_{i+s} = \left(\sum_{j=1}^{m} \tbinom{j+s-2}{s-1} r_j u_1^{j+s-1}\right) \cdot u_1^{i+1} + \cdots + \left(\sum_{j=k}^{m} \frac{\binom{j+s-k-1}{s-1} r_j u_1^{j+s-k}}{(k-1)!}\right) \cdot (i+k-1)\cdots(i+1)u_1^{i+k} + \cdots + \frac{r_m u_1^s}{(m-1)!} \cdot (i+m-1)\cdots(i+1)u_1^{i+m}. \tag{12}$$

Recall the Hankel matrices $H_t^{(n)}$ associated with coefficients of $\phi(z)$, $\{a_i\}_{i\in\mathbb{Z}_{\ge 0}}$:

$$H_t^{(n)} = \begin{bmatrix} a_n & \cdots & a_{n+t-1} \\ \vdots & \ddots & \vdots \\ a_{n+t-1} & \cdots & a_{n+2t-2} \end{bmatrix},$$

and consider a $t \times t$ Vandermonde-like matrix $\mathcal{W}$ associated with the representation basis in (11) and (12). For $i = 1, \ldots, t$, the i-th column of $\mathcal{W} = \mathcal{W}[1:t]$ is

$$\mathcal{W}[i] = \begin{bmatrix} u_1^{n+i} \\ (n+i)\cdot u_1^{n+i+1} \\ (n+i+1)(n+i)\cdot u_1^{n+i+2} \\ \vdots \\ (n+m+i-2)\cdots(n+i)\cdot u_1^{n+i+m-1} \\ u_{m+1}^{n+i} \\ \vdots \\ u_t^{n+i} \end{bmatrix},$$

that gives a basis for a_i in (9). The first m entries of the i-th column give a representation basis for σ_i in (11).

Based on Lemma 3 and the representation in (12), the associated Hankel system can be factorized as $H_t^{(n)} = \mathcal{T}\cdot\mathcal{W}$, in which $\mathcal{T}$ is a $t\times t$ matrix such that for $m \le t$, $k = 1, \ldots, m$, the k-th column in the first m columns of $\mathcal{T}[1:m]$ is

$$\mathcal{T}[k] = \begin{bmatrix} \frac{r_k}{(k-1)!} \\ \sum_{j=k}^{m} \frac{r_j}{(k-1)!} u_1^{j+1-k} \\ \vdots \\ \sum_{j=k}^{m} \frac{\binom{j-k+s-1}{s-1} r_j}{(k-1)!} u_1^{j+s-k} \\ \vdots \\ \sum_{j=k}^{m} \frac{\binom{j-k+t-2}{t-2} r_j}{(k-1)!} u_1^{j+t-1-k} \end{bmatrix},$$

where $1 \le s \le t-1$. The $(m+1)$-th to t-th columns of $\mathcal{T}$ form

$$\mathcal{T}[m+1:t] = \begin{bmatrix} r_{m+1} & \cdots & r_t \\ r_{m+1}u_{m+1} & \cdots & r_t u_t \\ \vdots & \ddots & \vdots \\ r_{m+1}u_{m+1}^{t-1} & \cdots & r_t u_t^{t-1} \end{bmatrix}.$$

Define $\overline{a}_{i+1} = \overline{\sigma}_{i+1} + r_{m+1}u_{m+1}^{i+2} + \cdots + r_t u_t^{i+2}$ with $\overline{\sigma}_{i+1} = r_1 u_1^{i+2} + r_2\binom{i+1}{1}u_1^{i+3} + \cdots + r_m\binom{i+m-1}{m-1}u_1^{i+m+1}$, then for

$$\overline{H}_t^{(n+1)} = \begin{bmatrix} \overline{a}_{n+1} & \overline{a}_{n+2} & \cdots & \overline{a}_{n+t} \\ \overline{a}_{n+2} & \overline{a}_{n+3} & \cdots & \overline{a}_{n+t+1} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{n+t} & \overline{a}_{n+t+1} & \cdots & \overline{a}_{n+2t-1} \end{bmatrix}$$

we have $\overline{H}_t^{(n+1)} = \mathcal{T}\cdot\mathcal{U}\cdot\mathcal{W}$ with

$$\mathcal{U} = \begin{bmatrix} u_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & u_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_{m+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & u_t \end{bmatrix}.$$

Therefore, $u_1, \ldots, u_1, u_{m+1}, \ldots, u_t$ are solutions to the generalized eigenvalue system $\overline{H}_t^{(n+1)}v = uH_t^{(n)}v$, and $z_1, \ldots, z_1, z_{m+1}, \ldots, z_t$ are solutions to z in

$$H_t^{(n)}v = z\overline{H}_t^{(n+1)}v. \tag{13}$$

Theorem 4. *Let function $f(z)$ be analytic at $z = 0$ and meromorphic in disk $D : |z| < \delta$, its poles $z_j = u_j^{-1}$, which may be finite or infinite in number, be numbered such that*

$$0 < |z_1| \le |z_2| \le \cdots \le |z_t| < \delta < |z_{t+1}|.$$

Each pole occurs as many times in the sequence $z_1, z_2, \ldots,$ as indicated by its order. (If f has only t poles, then $z_{t+1} = \infty$.)

Consider the Taylor expansion of $f(z) = \sum_{i=0}^{\infty}\tilde{a}_i z^i$, and denote the Hankel matrix associated with $\{\tilde{a}_i\}_{i\in\mathbb{Z}_{\ge 0}}$ as $\mathcal{H}_t^{(n)}$, then the t solutions of the generalized eigenvalue problem

$$\mathcal{H}_t^{(n)}v = z\mathcal{H}_t^{(n+1)}v \tag{14}$$

converge to the first t poles of $f(z)$: $z_1, z_2, \ldots, z_t$ as $n \to \infty$.

Proof. We prove for the case when $f(z)$ has a multiple pole of order $m > 1$ at z_1 and all other poles $z_{m+1}, \ldots, z_t$ simple. The generalization to various multiple pole combinations is straightforward.

Recall $\phi(z) = \sum_{i=0}^{\infty} a_i z^i$ and $g(z) = \sum_{i=0}^{\infty} b_i z^i$, the Taylor coefficients for $f(z) = \sum_{i=0}^{\infty}\tilde{a}_i z^i = \phi(z) + g(z)$ can be written as $\tilde{a}_i = a_i + b_i$.

Consider $\tilde{a}_{i+1} = \overline{a}_{i+1} + \gamma_{i+1} + b_i$, in which

$$\begin{aligned}
a_{i+1} =& r_1 u_1^{i+2} + r_2 \tbinom{i+2}{1} u_1^{i+3} + \cdots + r_m \tbinom{i+m}{m-1} u_1^{i+m+1} \\
&+ r_{m+1} u_{m+1}^{i+2} + \cdots + r_t u_t^{i+2} \\
=& r_1 u_1^{i+2} + r_2 \left[\tbinom{i+1}{1} + \tbinom{i+1}{1-1} \right] u_1^{i+3} + \cdots \\
&+ r_m \left[\tbinom{i+m-1}{m-1} + \tbinom{i+m-1}{m-2} \right] u_1^{i+m+1} \\
&+ r_{m+1} u_{m+1}^{i+1} + \cdots + r_t u_t^{i+1} \\
=& \overline{a}_{i+1} + \underbrace{\sum_{j=2}^{m} \tbinom{i+j-1}{j-2} r_j u_1^{i+j+1}}_{\gamma_{i+1}}.
\end{aligned}$$

Since $\lim_{n\to\infty} \binom{n+j-1}{j-2} / \binom{n+j-1}{j-1} = 0$ for $j = 1, \ldots, m$, for any $\epsilon/2 > 0$ there exists $\zeta_\gamma \in \mathbb{Z}_{\geq 0}$ such that for all $n > \zeta_\gamma$

$$\begin{aligned}
\frac{|\gamma_{n+1}|}{|\overline{a}_{n+1}|} &= \frac{|\gamma_{n+1}|}{\left| \sum_{j=1}^{m} r_j \binom{n+j-1}{j-1} u_1^{n+j+1} + \sum_{j=m+1}^{t} r_j u_j^{n+2} \right|} \\
&\leq \left| \frac{\left| \sum_{j=2}^{m} r_j \binom{n+j-1}{j-2} u_1^{j-1} \right|}{\left| \sum_{j=1}^{m} r_j \binom{n+j-1}{j-1} u_1^{j-1} \right| - \left| \sum_{j=m+1}^{t} r_j (\frac{u_j}{u_1})^{n+2} \right|} \right| < \frac{\epsilon}{2}.
\end{aligned}$$

On the other hand, by the Cauchy coefficient estimate, there is a number $\eta > 0$ such that $|b_n| \leq \eta \rho^n$ and $0 < \rho < |u_t| \leq |u_j| < |u_1|$ for $j = m+1, \ldots, t$, and

$$\lim_{n\to\infty} \left(\frac{|\rho|}{|u_1|} \right)^n \leq \lim_{n\to\infty} \left(\frac{|u_j|}{|u_1|} \right)^n = 0.$$

There also exists $\zeta_b \in \mathbb{Z}_{\geq 0}$ such that for all $n > \zeta_b$

$$\begin{aligned}
\frac{|b_n|}{|a_n|} &= \frac{|b_n|}{\left| \sum_{j=1}^{m} r_j \binom{n+j-1}{j-1} u_1^{n+j} + \sum_{j=m+1}^{t} r_j u_j^{n+1} \right|} \\
&\leq \left| \frac{\eta \cdot \left(\frac{\rho}{|u_1|} \right)^n}{\left| \sum_{j=1}^{m} r_j \binom{n+j-1}{j-1} u_1^{j} \right| - \left| \sum_{j=m+1}^{t} r_j u_j (\frac{u_j}{u_1})^{n} \right|} \right| < \frac{\epsilon}{2}.
\end{aligned}$$

For the same reason, $|b_{n+1}|/|\overline{a}_{n+1}| < \epsilon/2$ for all $n > \zeta_b$.

Let $B_t^{(n)}$ be the Hankel matrix associated with $\{b_i\}_{i\in\mathbb{Z}_{\geq 0}}$, $\Gamma_t^{(n)}$ the Hankel matrix associated with $\{\gamma_i\}_{i\in\mathbb{Z}_{\geq 0}}$, then

$$\begin{aligned}
\mathcal{H}_t^{(n)} &= H_t^{(n)} + B_t^{(n)}, \\
\mathcal{H}_t^{(n+1)} &= \overline{H}_t^{(n+1)} + \Gamma_t^{(n+1)} + B_t^{(n+1)}.
\end{aligned}$$

Let $\zeta = \max(\zeta_b, \zeta_\gamma)$, then for all $n > \zeta$, $B_t^{(n)}$ and $(\Gamma_t^{(n+1)} + B_t^{(n+1)})$ can be regarded as the ϵ-perturbations in the generalized eigenvalue problem

$$\begin{aligned}
&\overbrace{\left(H_t^{(n)} + B_t^{(n)} \right)}^{\mathcal{H}_t^{(n)}} (v + \epsilon v^{(1)} + \cdots) \\
&= (z + \epsilon z^{(1)} + \cdots) \underbrace{\left(\overline{H}_t^{(n+1)} + (\Gamma_t^{(n+1)} + B_t^{(n+1)}) \right)}_{\mathcal{H}_t^{(n+1)}} \\
&\times (v + \epsilon v^{(1)} + \cdots),
\end{aligned}$$

and its solution converges to $z_1, \ldots, z_1, z_{m+1}, \ldots, z_t$. □

The generating polynomial in the Ben-Or/Tiwari algorithm from computer algebra is a special case of the Hadamard polynomial when the meromorphic function is rational and all the poles are simple [11, also see Theorem 7.7f].

THEOREM 5. *Let function $f(z)$ be analytic at $z = 0$ and meromorphic in the disk $D : |z| < \delta$, its poles $z_j = u_j^{-1}$, which may be finite or infinite in number, be numbered such that*

$$0 < |z_1| \leq |z_2| \leq \cdots \leq |z_t| < \delta < |z_{t+1}|.$$

Each pole occurs as many times in the sequence $z_1, z_2, \ldots$, as indicated by its order. (If f has only t poles, then $z_{t+1} = \infty$.)

Consider the Taylor expansion of $f(z) = \sum_{i=0}^{\infty} \tilde{a}_i z^i$, and denote the Hankel matrix associated with $\{\tilde{a}_i\}_{i\in\mathbb{Z}_{\geq 0}}$ as $\mathcal{H}_t^{(n)}$. Let the polynomial $\Lambda_t^{(n)}(u) = u^t + \lambda_{t-1}^{(n)} u^{t-1} + \cdots + \lambda_0^{(n)}$ with coefficients formed from the solution of the Hankel system associated with $\{\tilde{a}_i\}_{i\in\mathbb{Z}_{\geq 0}}$

$$\underbrace{\begin{bmatrix} \tilde{a}_n & \tilde{a}_{n+1} & \ldots & \tilde{a}_{n+t-1} \\ \tilde{a}_{n+1} & \tilde{a}_{n+2} & \ldots & \tilde{a}_{n+t} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n+t-1} & \tilde{a}_{n+t} & \ldots & \tilde{a}_{n+2t-2} \end{bmatrix}}_{\mathcal{H}_t^{(n)}} \begin{bmatrix} \lambda_0^{(n)} \\ \lambda_1^{(n)} \\ \vdots \\ \lambda_{t-1}^{(n)} \end{bmatrix} = - \begin{bmatrix} \tilde{a}_{n+t} \\ \tilde{a}_{n+t+1} \\ \vdots \\ \tilde{a}_{n+2t-1} \end{bmatrix}. \tag{15}$$

Then the polynomial $\Lambda_t^{(n)}(u) = u^t + \lambda_{t-1}^{(n)} u^{t-1} + \cdots + \lambda_0^{(n)}$ converges to $\Lambda_t(u) = (u - u_1) \cdots (u - u_t)$, whose zeros are the reciprocals of the first t poles of $f(z)$, as $n \to \infty$.

PROOF. By Cramer's rule, solutions of (15) are coefficients in the corresponding Hadamard polynomial $\Lambda_t^{(n)}(u)$. Then apply [11, Theorem 7.7e]. □

In the hypotheses of Theorem 5, we consider the rational part $\phi(z) = \sum_{i=0}^{\infty} a_i z^i$ of $f(z)$ (also see (9)) and repeat the analogous lemmas.

LEMMA 4. *The Hankel system $H_i^{(n)}$ associated with sequence $\{a_i\}_{i\in\mathbb{Z}_{\geq 0}}$ is singular for all $i \geq t+1$.*

LEMMA 5. *The Hankel system $H_i^{(n)}$ associated with sequence $\{a_i\}_{i\in\mathbb{Z}_{\geq 0}}$ is non-singular for all $0 \leq i \leq t$.*

4. CONCLUSION

We show that the poles of a univariate meromorphic function $f(z)$, computed by the qd-algorithm, can be interpreted as the generalized eigenvalues for a pair of shifted Hankel matrices. As a special case, when all the poles are simple, such generalized eigenvalue problem can be viewed as a reformulation of Prony's method, a method that is closely related to the Ben-Or/Tiwari algorithm for sparse polynomial interpolation in computer algebra. Furthermore, the vanishing of e columns in the qd-algorithm is analogous to the zero discrepancies in the early termination strategy. We are currently investigating the numerical behaviors associated with these connections.

As the homogeneous multivariate qd-algorithm can be regarded as a parameterized univariate qd-algorithm [4, 5], our generalized eigenvalue interpretation can thus be extended to the homogeneous multivariate qd-algorithm as well.

Based on our interpretation and the subsequent connections, additional numerical methods for computing poles of a meromorphic function and sparse polynomial interpolation can be developed. The sensitivity and practicality of these approaches, especially in comparison with the existing algorithms, remain to be further studied and constitute our current research.

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