Extracting Numerical Factors of Multivariate Polynomials from Taylor Expansions

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ABSTRACT

We present a method to extract factors of multivariate polynomials with complex coefficients in floating point arithmetic. We establish the connection between the reciprocal of a multivariate polynomial and its Taylor expansion. Since the multivariate Taylor coefficients are determined by the irreducible factors of the given polynomial, we reconstruct the factors from the Taylor expansion. As each irreducible factor, regardless of its multiplicity, can be separately extracted, our method can lead toward the complete numerical factorization of multivariate polynomials.

Categories and Subject Descriptors

F.2.1 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity—*Numerical Algorithms and Problems*; G.1 [**Mathematics of Computing**]: Numerical Analysis; I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Algorithms*

General Terms

algorithms, theory

Keywords

polynomial factorization, numerical factorization, approximate factorization, partial fraction, qd-algorithm, Taylor expansion, multivariate Padé approximation, pole detection, symbolic-numeric method

1. INTRODUCTION

We consider the problem of computing an irreducible factor of a given multivariate polynomial in a finite precision environment. For a given polynomial f, we look at the connection between the factors of f and the Taylor expansion of its reciprocal $1/f$. While the univariate case has been extensively studied (e.g., see [8, pp. 596–597]), until recently the detailed analysis of the multivariate case is lacking. In this paper, we establish the connection between the irreducible factors of a multivariate polynomial $f(x_1, \ldots, x_n)$ and the Taylor expansion of its reciprocal $1/f(x_1, \ldots, x_n)$. Based on such connection, we present a method that can reconstruct the irreducible factors of $f(x_1, \ldots, x_n)$ from the associated Taylor expansion.

A multivariate polynomial is a product of finitely many irreducible factors. Recovering each individual factor eventually leads to the computation of all factors and hence the complete factorization of the given multivariate polynomial.

The problem of factoring multivariate polynomials in complex floating point arithmetic was recognized by Kaltofen in 1985 when he gave one of the first polynomial-time algorithms to factorize multivariate polynomials exactly [11]. The idea to tackle the inexact case as an optimization problem was independently suggested by Sasaki et al. [20] and Kaltofen [12].

With a renewed interest, over the past ten years, a significant body of results in numerical, as well as approximate, multivariate polynomial factorization have been achieved (see, e.g., [6, 9, 10, 2, 4, 19, 1, 21, 7, 5, 15] and the references given there). A related problem is to bound a polynomial away from irreducible polynomials, for which we refer to [17, 14, 18]. Such accomplishments undoubtedly contribute to the recent prominence of symbolic-numeric computation and approximate algebra.

Our method is a numerical approach by nature because it is built upon the convergence behavior of the multivariate Taylor expansion. The novelty of our approach is threefold. First, compared to existing factorization algorithms, our method reconstructs a factor at a time. Second, we do not require the input polynomial to be square-free. Third, our method is suitable for iterative improvement. In other words, under the constraint of finite precision, when more computational effort is spent, the accuracy increases.

In addition, the construction of a multivariate partial fraction decomposition in a monomial order is embedded in our theoretical development (for multivariate partial fractions implemented in certain computer algebra systems, we refer to [22]).

The rest of the paper is organized as follows. Section 2 establishes the theoretical connection between a multivariate polynomial and the Taylor expansion of its reciprocal. Based on the established connection, Section 3 shows how to extract factors of a given polynomial in various circumstances. Then in Section 4, we comment on our results and conclude with current research directions in complete factorization and irreducibility testing.

2. TAYLOR EXPANSION AND FACTORS

In this section we explain the connection between the factors of a polynomial and the Taylor expansion of its reciprocal. The univariate case is repeated in §2.1. The multivariate connection for a single irreducible factor, which underpins our computational method, is presented in §2.2. In §2.3 we discuss the situation when there exist multiple factors in the given multivariate polynomial.

2.1 Univariate Case

Suppose $f(x)$ is a univariate polynomial. For simplicity, let $f(x) = (1-b_1x) \cdots (1-b_tx) \in \mathbb{C}[x]$ with $|b_1| > \cdots > |b_t|$. Consider the Taylor expansion of its reciprocal

$$\begin{aligned}\frac{1}{f(x)} &= \frac{r_1}{1-b_1x} + \cdots + \frac{r_t}{1-b_tx} \\ &= r_1(1 + b_1x + b_1^2x^2 + \cdots) + \cdots \\ &\quad + r_t(1 + b_tx + b_t^2x^2 + \cdots) = \sum_{i=0}^{\infty} a_ix^i \qquad (1)\end{aligned}$$

with $r_1, \ldots, r_t \in \mathbb{C}$.

By recollecting the coefficients in (1), the i-th Taylor coefficient a_i is a sum of exponentials. For $i = 0, 1, 2, \ldots,$

$$a_i = r_1b_1^i + \cdots + r_tb_t^i. \qquad (2)$$

Consider the sequence $\{a_i\}_{i\geq 0}$ formed by the Taylor coefficients a_i. From (2) we can conclude that the sequence $\{a_i\}_{i\geq 0}$ is linearly generated and the associated generating polynomial is $\Lambda(z) = (z-b_1)\cdots(z-b_t)$. Further details are referred to, e.g., [16].

Note that $|b_1| > \cdots > |b_t|$. As $i \to \infty$, the exponential term $r_1b_1^i$ dominates a_i. Such property can be exploited to recover the zeros of $f(x)$ in an order that reflects their moduli [8, pp. 617–618].

As for the more general case of $|b_1| \geq \cdots \geq |b_t|$, where $|b_i| = \cdots = |b_j|$ for some $1 \leq i < j \leq t$, we refer to the treatment in [8, §7.9].

2.2 Multivariate: Single Irreducible Factor

We investigate the multivariate case. Consider an irreducible polynomial $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$. Assume f has a non-zero constant term since this can always be achieved by shifting the basis in the representation of f. For the time being we assume for simplicity that $f(0, \ldots, 0) \neq 0$ and hence that the Taylor series in the sequel are all considered around the origin.

Let the constant of $f(x_1, \ldots, x_n)$ be normalized to 1, that is, $f(x_1, \ldots, x_n) = 1 - p(x_1, \ldots, x_n)$ with $p(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ having zero constant. Expand $1/f$ into a geometric series,

$$\begin{aligned}\frac{1}{f} &= \frac{1}{1-p} = 1 + p + p^2 + \cdots \\ &= \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n_{\geq 0}} a_{i_1,\ldots,i_n}x_1^{i_1}\cdots x_n^{i_n} = \sum_{\vec{\imath}\in\mathbb{Z}^n_{\geq 0}} a_{\vec{\imath}}x^{\vec{\imath}}, \qquad (3)\end{aligned}$$

in which $a_{\vec{\imath}}x^{\vec{\imath}} = a_{i_1,\ldots,i_n}x_1^{i_1}\cdots x_n^{i_n}$ and $\vec{\imath} = (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$.

The expression (3) may appear similar to (1). However, in the univariate case the Taylor coefficients can be directly expressed in terms of the linear factors in the denominator, while in the multivariate case (3), the denominator $1 - p(x_1, \ldots, x_n)$ is an irreducible multivariate polynomial. The multivariate Taylor coefficients $a_{\vec{\imath}}$ are not directly collected in the powers of p, but with respect to the power standard basis $x^{\vec{\imath}} = x_1^{i_1}\cdots x_n^{i_n}$. Our aim is to explore the multivariate Taylor coefficients $a_{\vec{\imath}}$ in relation to the polynomial $1 - p(x_1, \ldots, x_n)$.

Since $p(x_1, \ldots, x_n)$ is assumed to have constant zero, let

$$p(x_1, \ldots, x_n) = b_1x^{\vec{d}_1} + b_2x^{\vec{d}_2} + \cdots + b_mx^{\vec{d}_m}, \qquad (4)$$

in which $x^{\vec{d}_i}$ denotes $x_1^{d_{i1}}\cdots x_n^{d_{in}}$ for $i = 1, \ldots, m$. Let $x^{\vec{d}_0} = x_1^0\cdots x_n^0 = 1$. We require $b_m \neq 0$ and $x^{\vec{d}_1}, \ldots, x^{\vec{d}_m}$ to follow a monomial order $\prec$ in $\mathbb{C}[x_1, \ldots, x_n]$ such that[1]

$$\begin{aligned}&\operatorname{multideg}(x^{\vec{d}_0}) \prec \operatorname{multideg}(x^{\vec{d}_1}) \prec \operatorname{multideg}(x^{\vec{d}_2}) \prec \cdots \\ &\qquad \cdots \prec \cdots \prec \operatorname{multideg}(x^{\vec{d}_m}) \prec \cdots. \qquad (5)\end{aligned}$$

In general, a monomial order does not guarantee that each monomial can be enumerated from $(0, \ldots, 0)$. For example, in $\mathbb{C}[x_1, x_2]$ under a lexicographic order, monomial x_1 can never be enumerated from $(0, 0)$ because for all $n \geq 0$, $\operatorname{multideg}(x_2^n) \prec_{\text{lex}} \operatorname{multideg}(x_1)$. From now on, in our discussion we restrict all monomial orders to enumerate any monomial in $\mathbb{C}[x_1, \ldots, x_n]$ under consideration.

Let $A_i = p^i$. We consider the sequence of multivariate polynomials $\{A_i = p^i\}_{i\geq 0}$. By definition, the sequence $\{A_i\}_{i\geq 0}$ is generated by the linear recurrence equation $A_{i+1} = p \cdot A_i$ that can be represented by an associated generating polynomial $\Lambda(p) = 1 - p$. Because $p(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$, the generating polynomial $\Lambda(p)$ is a multivariate polynomial in $\mathbb{C}[x_1, \ldots, x_n]$. We remark that if p is regarded as an indeterminate, then $\Lambda(p)$ can be viewed as a univariate polynomial in p. That is, $\Lambda(p) \in \mathbb{C}[p]$.

Substitute (4) in the linear recurrence $A_{i+1} = p \cdot A_i$,

$$A_{i+1} = A_ib_1x^{\vec{d}_1} + \cdots + A_ib_mx^{\vec{d}_m}. \qquad (6)$$

Suppose $M, N \in \mathbb{Z}_{>0}$ and $M < N$. By repeating (6) for $i = M, M+1, \ldots, N-1, N$, we form the following system:

$$\begin{aligned}A_N &= A_{N-1}b_1x^{\vec{d}_1} + \cdots + A_{N-1}b_mx^{\vec{d}_m} \\ A_{N-1} &= A_{N-2}b_1x^{\vec{d}_1} + \cdots + A_{N-2}b_mx^{\vec{d}_m} \\ \vdots &= \qquad\vdots \\ A_{M+1} &= A_Mb_1x^{\vec{d}_1} + \cdots + A_Mb_mx^{\vec{d}_m} \\ A_M &= A_{M-1}b_1x^{\vec{d}_1} + \cdots + A_{M-1}b_mx^{\vec{d}_m}. \qquad (7)\end{aligned}$$

Now we are ready to link the polynomial $1 - p = 1 - b_1x^{\vec{d}_1} - \cdots - b_mx^{\vec{d}_m}$ to the multivariate Taylor coefficients $a_{\vec{\imath}}$ from (3). But for later reference we first elaborate in Theorems 1 and 2 a few details regarding the system (7).

According to (3), any $a_{\vec{\imath}}x^{\vec{\imath}}$ from the Taylor expansion can be collected from the infinite series $1+p+\cdots = A_0+A_1+\cdots$. But Theorem 1 states that it is sufficient to collect $a_{\vec{\imath}}x^{\vec{\imath}}$ from a finite partial sum $A_M + \cdots + A_N$ of the infinite series $A_0 + A_1 + \cdots$.

[1]See, e.g., [3, pp. 54–60] for the definition and further discussions of a monomial order.

THEOREM 1. *Under a monomial order $\prec$, for any $a_{\vec{k}}x^{\vec{k}}$ in* (3) *such that* $\text{multideg}(a_{\vec{k}}x^{\vec{k}}) \succeq \text{multideg}(p) = \text{multideg}(A_1)$, *there exist $M, N \in \mathbb{Z}_{>0}$ such that $M < N$ and $a_{\vec{k}}x^{\vec{k}}$ can be obtained from collecting either side of* (7).

PROOF. Since $\text{multideg}(a_{\vec{k}}x^{\vec{k}}) \succeq \text{multideg}(A_1)$, at least we have $M = 1$ that satisfies (7) and the existence of M is proved.

We use $\omega(p)$ to denote the non-zero term with the lowest multivariate degree in p. Because p has zero constant, $\text{multideg}(\omega(p)) \succ \text{multideg}(x^{\vec{d}_0}) = (0, \dots, 0)$, and

$$\begin{aligned}\text{multideg}(\omega(A_{i+1})) &= \text{multideg}(\omega(p^{i+1}))\\ &= \text{multideg}(\omega(p^i)) + \text{multideg}(\omega(p))\\ &\succ \text{multideg}(\omega(p^i)) = \text{multideg}(\omega(A_i)).\end{aligned}$$

Therefore, as $i \to \infty$, the monomial order of $\omega(A_i)$ is strictly increasing. For a given $a_{\vec{k}}x^{\vec{k}}$ there exists $N \in \mathbb{Z}_{>0}$ such that

$$\begin{aligned}\text{multideg}(a_{\vec{k}}x^{\vec{k}}) &\preceq \text{multideg}(\omega(A_N))\\ &\prec \text{multideg}(\omega(A_{N+1})) \prec \cdots .\end{aligned}$$

We conclude that $A_{N+1} + \cdots$ does not contribute to $a_{\vec{k}}x^{\vec{k}}$. Hence $a_{\vec{k}}x^{\vec{k}}$ can be collected from $A_M + \cdots + A_N$ alone. □

Theorem 2 further insists that M and N, which are used to define a finite partial sum for collecting a given $a_{\vec{\imath}}$, can simultaneously increase to infinity as $\text{multideg}(a_{\vec{\imath}}x^{\vec{\imath}})$ increases.

THEOREM 2. *For any $M > 0$, there exists $\vec{k} \in \mathbb{Z}^n_{\geq 0}$ such that for any $a_{\vec{\imath}}x^{\vec{\imath}}$ satisfying* $\text{multideg}(a_{\vec{\imath}}x^{\vec{\imath}}) \succeq \text{multideg}(a_{\vec{k}}x^{\vec{k}})$, *$a_{\vec{\imath}}x^{\vec{\imath}}$ can be collected from $A_M + \cdots + A_N$ for an $N \in \mathbb{Z}_{>0}$.*

In other words, if $\text{multideg}(a_{\vec{\imath}}x^{\vec{\imath}})$ *follows a monomial order that enumerates every term, then $a_{\vec{\imath}}x^{\vec{\imath}}$ can be collected from a finite partial sum $A_M + \cdots + A_N$ and both $M, N \to \infty$.*

PROOF. Since $N \to \infty$ is obvious, we only show $M \to \infty$. Because our monomial order enumerates all terms, for any $M > 0$ there exists $\vec{k} = (k_1, \dots, k_n)$ such that $\text{multideg}(a_{\vec{k}}x^{\vec{k}}) \succeq \text{multideg}(A_M)$. Since $\text{multideg}(A_M) \succ \text{multideg}(A_{M-1}) \succ \cdots$, the sum $A_0 + \cdots + A_{M-1}$ does not contain a term with monomial $x^{\vec{k}}$ or of a higher multivariate degree. As a result, if $\text{multideg}(a_{\vec{\imath}}x^{\vec{\imath}}) \succeq \text{multideg}(a_{\vec{k}}x^{\vec{k}})$, $a_{\vec{\imath}}x^{\vec{\imath}}$ can be collected from $A_M + A_{M+1} + \cdots + A_N$ and both $M, N \to \infty$. □

Theorems 1 and 2 form the foundation of our convergence arguments: as $\text{multideg}(a_{\vec{\imath}}x^{\vec{\imath}})$ increases, $a_{\vec{\imath}}x^{\vec{\imath}}$ can be solely captured from higher powers of p in (3), which better reflect the dominating properties.

Return to the link between the irreducible polynomial $1-p$ and the multivariate Taylor coefficients $a_{\vec{k}}$. We collect $a_{\vec{k}}x^{\vec{k}}$ from both sides of (7),

$$\begin{aligned}a_{\vec{k}}x^{\vec{k}} = a_{\vec{k}-\vec{d}_1}x^{\vec{k}-\vec{d}_1} \cdot b_1x^{\vec{d}_1} + a_{\vec{k}-\vec{d}_2}x^{\vec{k}-\vec{d}_2} \cdot b_2x^{\vec{d}_2} + \cdots\\ + a_{\vec{k}-\vec{d}_m}x^{\vec{k}-\vec{d}_m} \cdot b_mx^{\vec{d}_m}\end{aligned}$$

resulting in

$$a_{\vec{k}} = b_1a_{\vec{k}-\vec{d}_1} + b_2a_{\vec{k}-\vec{d}_2} + \cdots + b_ma_{\vec{k}-\vec{d}_m}. \tag{8}$$

We repeat (8) for various $a_{\vec{k}_j}, \dots, a_{\vec{k}_{j+m-1}}$, m times in total, and form a linear system

$$\begin{bmatrix} a_{\vec{k}_j-\vec{d}_1} & \cdots & a_{\vec{k}_j-\vec{d}_m}\\ \vdots & \ddots & \vdots\\ a_{\vec{k}_{j+m-1}-\vec{d}_1} & \cdots & a_{\vec{k}_{j+m-1}-\vec{d}_m}\end{bmatrix}\begin{bmatrix} b_1\\ \vdots\\ b_m\end{bmatrix} = \begin{bmatrix} a_{\vec{k}_j}\\ \vdots\\ a_{\vec{k}_{j+m-1}}\end{bmatrix}. \tag{9}$$

THEOREM 3. *Let $x^{\vec{d}_\ell}$ follow the monomial order in* (5), *then the $\ell+1$ by $\ell+1$ matrix*

$$\Delta^{(j)}_{\ell+1} = \begin{bmatrix} a_{\vec{k}_j} & a_{\vec{k}_j-\vec{d}_1} & \cdots & a_{\vec{k}_j-\vec{d}_\ell}\\ \vdots & \vdots & \ddots & \vdots\\ a_{\vec{k}_{j+\ell-1}} & a_{\vec{k}_{j+\ell-1}-\vec{d}_1} & \cdots & a_{\vec{k}_{j+\ell-1}-\vec{d}_\ell}\\ a_{\vec{k}_{j+\ell}} & a_{\vec{k}_{j+\ell}-\vec{d}_1} & \cdots & a_{\vec{k}_{j+\ell}-\vec{d}_\ell}\end{bmatrix} \tag{10}$$

is singular for $\ell = m, m+1, m+2, \dots$.

PROOF. Because of (8), the first column $[a_{\vec{k}_j}, \dots, a_{\vec{k}_{j+\ell}}]^{\text{Tr}}$ is a linear combination of the m consecutive columns in $\Delta^{(j)}_{\ell+1}$. □

We look at the singularity of $\Delta^{(j)}_{\ell+1}$ for $\ell < m$. In $p = b_1x^{\vec{d}_1} + \cdots + b_\ell x^{\vec{d}_\ell} + \cdots + b_mx^{\vec{d}_m}$, the coefficient of the leading term is non-zero, which means $b_m \neq 0$ and $\text{multideg}(x^{\vec{d}_m}) \succ \cdots \succ \text{multideg}(x^{\vec{d}_\ell}) \succ \cdots \succ \text{multideg}(x^{\vec{d}_1})$. But it is possible that some b_ℓ for $1 \leq \ell < m$ are zero and these zero terms can cause $\Delta^{(j)}_{\ell+1}$ to be singular for $1 \leq \ell < m$.

Example 1. Let $p = x_1^5 + x_2^2$ and $a_{\vec{\imath}}$ be the coefficients in the expansion

$$\begin{aligned}&1 + p + p^2 + p^3 + \cdots\\ &\quad = a_{0,0} + a_{1,0}x_1 + a_{0,1}x_2 + a_{2,0}x_1^2 + a_{1,1}x_1x_2 + \cdots .\end{aligned}$$

Let the enumeration of $a_{\vec{k}_\ell}x^{\vec{k}_\ell}$ follow the monomial order of 1, x_1, x_2, x_1^2, x_1x_2, x_2^2, ... The zero terms in p can cause zero terms in the expansion of $1 + p + p^2 + \cdots$. The corresponding matrix $\Delta^{(j)}_{\ell+1}$ can be singular for $\ell < m$. For example, the matrix

$$\begin{aligned}\Delta^{(199)}_{1+1} &= \begin{bmatrix} a_{\vec{k}_{199}} & a_{\vec{k}_{199}-\vec{d}_1}\\ a_{\vec{k}_{200}} & a_{\vec{k}_{200}-\vec{d}_1}\end{bmatrix} = \begin{bmatrix} a_{\vec{k}_{199}} & a_{\vec{k}_{199}-(1,0)}\\ a_{\vec{k}_{200}} & a_{\vec{k}_{200}-(1,0)}\end{bmatrix}\\ &= \begin{bmatrix} a_{10,9} & a_{9,9}\\ a_{9,10} & a_{8,10}\end{bmatrix} = \begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix} \text{ is singular}\end{aligned}$$

when $\ell = 1 < m = 16$ and $b_mx^{\vec{d}_m} = b_{16}x^{\vec{d}_{16}}$ is the leading term in p. ⊠

If all $b_1, \dots, b_m$ are known to be non-zero, or if we consider the expression that only includes the non-zero terms in p,

$$p = \beta_1x^{\vec{\delta}_1} + \cdots + \beta_\ell x^{\vec{\delta}_\ell} + \cdots + \beta_\mu x^{\vec{\delta}_\mu},\ \beta_\ell \neq 0, \tag{11}$$

for $\ell = 1, \dots, \mu$, then we can conclude the non-singularities for the associated matrices $\bar{\Delta}^{(v)}_{\ell+1}$.

THEOREM 4. *Suppose $x^{\vec{\delta}_1}, \dots, x^{\vec{\delta}_\ell}, \dots, x^{\vec{\delta}_\mu}$ in* (11) *follow a monomial order. Let $a_{\vec{k}_v}x^{\vec{k}_v}, \dots, a_{\vec{k}_{v+\ell}}x^{\vec{k}_{v+\ell}}$ correspond* *to the terms formed by the expansion of these non-zero terms* $1 + p + p^2 + \cdots$, *then the matrix*

$$\bar{\Delta}^{(v)}_{\ell+1} = \begin{bmatrix} a_{\vec{k}_v} & a_{\vec{k}_v - \vec{\delta}_1} & \cdots & a_{\vec{k}_v - \vec{\delta}_\ell} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\vec{k}_{v+\ell}} & a_{\vec{k}_{v+\ell} - \vec{\delta}_1} & \cdots & a_{\vec{k}_{v+\ell} - \vec{\delta}_\ell} \end{bmatrix} \tag{12}$$

is non-singular for $\ell = 0, 1, \ldots, \mu - 1$.

On the other hand, if all possible terms in p *are known to be non-zero, that is* $b_\ell \neq 0$ *for* $\ell = 1, \ldots, m$, *then the matrix* $\Delta^{(j)}_{\ell+1}$ *is non-singular for* $\ell = 1, \ldots, m - 1$.

Proof. Removing zero terms in (8), we have

$$a_{\vec{k}} = \beta_1 a_{\vec{k}-\vec{\delta}_1} + \beta_2 a_{\vec{k}-\vec{\delta}_2} + \cdots + \beta_\mu a_{\vec{k}-\vec{\delta}_\mu}, \tag{13}$$

with all $\beta_\ell \neq 0$. Apply (13) to all corresponding entries in $\bar{\Delta}^{(v)}_{\ell+1}$. Each entry is, at least asymptotically, a sum of μ exponentials. The dimension of matrix (12) reflects the exponents of these μ exponentials. Therefore $\bar{\Delta}^{(v)}_{\ell+1}$ is non-singular if its dimension is less than $\mu + 1$.[2] □

Lemma 1. *Suppose* $x^{\vec{\delta}_1}, \ldots, x^{\vec{\delta}_\ell}, \ldots, x^{\vec{\delta}_\mu}$ *in (11) follow a monomial order. Let* $a_{\vec{k}_v} x^{\vec{k}_v}, \ldots, a_{\vec{k}_{v+\ell}} x^{\vec{k}_{v+\ell}}$ *correspond to the terms formed by the expansion of these non-zero terms* $1 + p + p^2 + \cdots$, *then the matrix*

$$\bar{\Delta}^{(v)}_{\mu+1} = \begin{bmatrix} a_{\vec{k}_v} & a_{\vec{k}_v - \vec{\delta}_1} & \cdots & a_{\vec{k}_v - \vec{\delta}_\mu} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\vec{k}_{v+\mu}} & a_{\vec{k}_{v+\mu} - \vec{\delta}_1} & \cdots & a_{\vec{k}_{v+\mu} - \vec{\delta}_\mu} \end{bmatrix} \tag{14}$$

is singular with a rank deficient by 1.

Similarly, if all possible terms in p *are known to be non-zero, that is* $b_\ell \neq 0$ *for* $\ell = 1, \ldots, m$, *then the matrix* $\Delta^{(j)}_{m+1}$ *is singular with a rank deficient by 1.*

Proof. Apply Theorem 3 to the polynomial $p(x_1, \ldots, x_n)$ represented as (11). □

Theorem 4 and Lemma 1 can be used to detect the leading term of p: when $\bar{\Delta}^{(v)}_{\ell+1}$ or $\Delta^{(j)}_{m+1}$ first becomes singular, then $\ell = \mu$ or $\ell = m$.

Nevertheless, both Theorem 4 and Lemma 1 require the knowledge of the non-zero terms in $p(x_1, \ldots, x_n)$, which is normally not supplied. By shifting and rotating the representation basis, we can always make all possible terms in a given polynomial non-zero.

2.3 Multivariate: Several Irreducible Factors

In §2.2 we establish the connection between a multivariate irreducible polynomial and the Taylor expansion of its reciprocal. Now we look at the situation when the given multivariate polynomial is a product of irreducible polynomials. Our results in §2.2 can still be applied to each individual factor.

Consider a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ whose constant is normalized to 1 and has the nontrivial factorization

$$f = 1 - p = (1 - p_1)(1 - p_2) \cdots (1 - p_t) \text{ for } t > 1, \tag{15}$$

[2]In the univariate case such matrix is comparable to a Hankel system whose entries are also sums of exponentials (see, e.g., [8, Theorem 7.5e] or [13, Theorem 4] in different contexts).

in which $p, p_1, \ldots, p_t$ are all assumed to have zero constant term and $1 - p_j$ is irreducible for all $1 \leq j \leq t$. We do not require $p_1, \ldots, p_t$ to be all distinct. In other words, f may have repeated factors.

Let

$$D_j = \prod_{i \neq j} (1 - p_i) \text{ for } j = 1, \ldots, t.$$

Consider

$$1 = R_1 D_1 + \cdots + R_t D_t + R, \tag{16}$$

for some multivariate polynomials $R, R_1, \ldots, R_t \in \mathbb{C}[x_1, \ldots, x_n]$. Note that for a set of given $D_1, \ldots, D_t$, there always exist (infinitely many combinations of) $R, R_1, \ldots, R_t \in \mathbb{C}[x_1, \ldots, x_n]$ such that (16) holds.

Substitute (16) into the numerator of $1/f$,

$$\begin{aligned} \frac{1}{f} = \frac{1}{1-p} &= \frac{R_1 D_1 + \cdots + R_t D_t + R}{(1-p_1)\cdots(1-p_t)} \\ &= \frac{r_1}{1-p_1} + \cdots + \frac{r_t}{1-p_t} + \frac{r}{(1-p_1)\cdots(1-p_t)} + q \end{aligned} \tag{17}$$

and $r_j, p_j, r, q \in \mathbb{C}[x_1, \ldots, x_n]$. Under a monomial order, we require that each term in r is not divisible by the leading term of $(1 - p_1) \cdots (1 - p_t)$, and for $j = 1, \ldots, t$ each term in r_j is not divisible by the leading term of p_j.

Since there can be many possible combinations of R, R_1, $\ldots$, R_t in (16), the corresponding $r_1, \ldots, r_t, r, q$ in (17) are not uniquely determined. However, for $j = 1, \ldots, t$, the numerators r_j and r cannot be zero at the same time. Otherwise, suppose $r_1 = r = 0$, then $1 - p_1$ is not a pole of $1/f$. It is worth mentioning that while $r_1, \ldots, r_t, r$ and q can vary, only the factors $1 - p_1, \ldots, 1 - p_t$ play a vital role in our convergence analysis.

For each p_j in (15), let $A^{(j)}_i = p^i_j$, then $1/f(x_1, \ldots, x_n)$ in (17) can be expanded as

$$\begin{aligned} \frac{1}{f} &= r_1(1 + p_1 + \cdots) + r_2(1 + p_2 + \cdots) + \cdots \\ &\quad + r_t(1 + p_t + \cdots) + r \prod_{j=1}^{t} (1 + p_j + \cdots) + q \\ &= r_1 \left(A^{(1)}_0 + A^{(1)}_1 + \cdots \right) + r_2 \left(A^{(2)}_0 + A^{(2)}_1 + \cdots \right) \\ &\quad + \cdots + r_t \left(A^{(t)}_0 + A^{(t)}_1 + \cdots \right) \\ &\quad + r \prod_{j=1}^{t} \left(A^{(j)}_0 + A^{(j)}_1 + \cdots \right) + q = \sum_{\vec{\imath} \in \mathbb{Z}^n_{\geq 0}} a_{\vec{\imath}} x^{\vec{\imath}}. \end{aligned} \tag{18}$$

Recall Theorem 2 in §2.2. When $f = 1 - p$ is irreducible, the Taylor coefficients $a_{\vec{\imath}}$ are determined by a finite sum of $p^M + p^{M+1} + \cdots + p^N$ and both $M, N \to \infty$ as $\text{multideg}(a_{\vec{\imath}} x^{\vec{\imath}})$ increases. Theorem 5 is a similar statement for a multivariate polynomial f when it is a product of irreducible polynomial factors.

Theorem 5. *For any* $M > 0$, *there exists* $\vec{k} \in \mathbb{Z}^n_{\geq 0}$ *such that for any* $a_{\vec{\imath}} x^{\vec{\imath}}$ *satisfying* $\text{multideg}(a_{\vec{\imath}} x^{\vec{\imath}}) \succeq \text{multideg}(a_{\vec{k}} x^{\vec{k}})$,

$a_{\vec{i}}x^{\vec{i}}$ can be collected from $S_{M,N}$ for an $N \in \mathbb{Z}_{>0}$ and

$$\begin{aligned} S_{M,N} = {} & r_1\left(A_M^{(1)} + \cdots + A_N^{(1)}\right) + r_2\left(A_M^{(2)} + \cdots + A_N^{(2)}\right) \\ & + \cdots + r_t\left(A_M^{(t)} + \cdots + A_N^{(t)}\right) \\ & + r\left(\underbrace{A_{M_1}^{(1)} \cdots A_{M_t}^{(t)} + \cdots + A_{N_1}^{(1)} \cdots A_{N_t}^{(t)}}_{T_{M,N}}\right), \quad (19) \end{aligned}$$

in which $M \leq M_1 + \cdots M_t$ and $N_1 + \cdots + N_t \leq N$.

In other words, if $\text{multideg}(a_{\vec{i}}x^{\vec{i}})$ follows a monomial order that enumerates every term, then $a_{\vec{i}}x^{\vec{i}}$ can be collected from a finite partial sum $S_{M,N}$ and both $M, N \to \infty$.

Proof. The polynomial q in (18) does not play a role in the convergence because it only has a finite number of terms that can contribute to the Taylor expansion.

For the rest of (18), apply Theorem 2 to each $r_j(A_0^{(j)} + A_1^{(j)} + \cdots)$ and $r\prod_{j=1}^{t}(A_0^{(j)} + A_1^{(j)} + \cdots)$ and sum them up. □

Theorem 5 extends a convergence argument in §2.2 to $f = (1 - p_1) \cdots (1 - p_t)$. As $\text{multideg}(x^{\vec{i}})$ increases following a monomial order, the Taylor coefficient $a_{\vec{i}}$ can be captured from a finite partial sum $S_{M,N}$ determined by higher powers and mixed powers of $p_1, \ldots, p_t$.

We conclude with three possible scenarios on the convergence behavior of $S_{M,N}$, as $M, N \to \infty$, in relation to the multivariate Taylor coefficients $a_{\vec{i}}$. Further details and related examples are treated in §3.

Scenario 1: Suppose there exists an irreducible factor in $f = (1 - p_1) \cdots (1 - p_t)$, say $1 - p_1$, such that p_1 dominates $S_{M,N}$ at convergence. Then such polynomial factor $1 - p_1$ is called *the dominating factor.* This case is comparable to §2.2 when f itself is an irreducible polynomial. As $\text{multideg}(a_{\vec{i}}x^{\vec{i}})$ increases, $a_{\vec{i}}$ reflects the dominating properties of higher powers of p_1.

Scenario 2: If there does not exist a dominating factor in $f = (1-p_1) \cdots (1-p_t)$, it is still possible that a subsequence of Taylor coefficients $a_{\vec{i}}$ is dominated by an irreducible factor, say $1 - p_1$, at convergence. In other words, there exists a subset $\mathsf{K} \subset \mathbb{Z}^n_{\geq 0}$ with cardinality $\#(\mathsf{K}) = \infty$ such that if $\text{multideg}(x^{\vec{k}})$ increases and $\vec{k} \in \mathsf{K} \subset \mathbb{Z}^n_{\geq 0}$, then the higher powers of p_1 dominate the convergence behavior of $\{a_{\vec{k}}\}_{\vec{k} \in \mathsf{K}}$. We call such factor *a partially dominating factor.*

Scenario 3: Suppose that each factor $1-p_j$, for $j = 1, \ldots, t$, neither dominates nor partially dominates. Instead, a product $(1 - p_1) \cdots (1 - p_s)$ with $s \geq 2$ dominates the remaining factors. Then each of the $1 - p_j$, $j = 1, \ldots, s$ is call *a non-dominating factor.*

We point out that the situation described in Scenario 3 is the multivariate analogue of the case $|b_1| = \cdots = |b_s|$ with $s \geq 2$ in the univariate §2.1. The situation described in Scenario 1 is comparable to the case $|b_1| > |b_2| > \cdots > |b_t|$. The case described in Scenario 2 is unique to the multivariate situation.

3. EXTRACTING NUMERICAL FACTORS

We use examples to illustrate how to extract irreducible factors in the three possible scenarios characterized above. Our strategy is based on the connection between the irreducible factors and the associated Taylor expansion.

To prepare our discussion, we define a *full-termed polynomial.*

Full-termed polynomial. Under a monomial order, we say a polynomial f is *full-termed* if all its possible terms with multivariate degrees up to $\text{multideg}(f)$ have non-zero coefficients.

If a polynomial is not full-termed, by shifting and rotating the representation basis, the same polynomial can be written as a full-termed polynomial in the new representation. For example, $f(x_1, x_2) = 1 + x_1^2 - x_2^3$ does not have all terms, but f can be represented as a full-termed polynomial in the power basis of

$$y_1 = -1 + \frac{3}{2}x_1 + \frac{1}{2}x_2, \quad y_2 = 3 - \frac{1}{2}x_1 - \frac{1}{2}x_2,$$

which results in a full-termed representation of the same polynomial

$$\begin{aligned} f = {} & -123 + 77y_1 + 227y_2 - 14y_1^2 - 88y_1y_2 - 134y_2^2 \\ & + y_1^3 + 9y_1^2y_2 + 27y_1y_2^2 + 27y_2^3. \end{aligned}$$

For a given polynomial, each factor can be represented as a full-termed polynomial under a (rotated and shifted) representation. The purpose of this paper is to demonstrate that each factor of f can be extracted. Without loss of generality, in the following discussion we assume that each factor is already a full-termed polynomial.

3.1 Dominating Factor

The basic case follows Scenario 1 at the end of §2.3. Let $f = (1-p_1) \cdots (1-p_t)$. As (18) shows, the coefficients $a_{\vec{i}}$ in the Taylor expansion of $1/f$ depend on the powers and the mixed powers of $p_1, \ldots, p_t$.

To study the convergence behavior, we focus on the situation when f is a product of two irreducible factors. That is, $f = (1 - p_1)(1 - p_2)$, and $1 - p_1$ is the dominating factor.[3] This happens when the powers of p_1 contribute more than the powers of p_2, in each variable, toward the Taylor coefficient $a_{\vec{i}}$ in the expansion (18) of $1/f$. For example, if $f = (1 - 5.2x_1 + 4.1x_2)(1 - x_1 - 1.1x_2^3)$ then $1 - 5.2x_1 + 4.1x_2$ is the dominating factor of f.

Due to the powering effect, the contributions, as well as the difference of the contributions, from p_1 and p_2 are exponential as $\text{multideg}(a_{\vec{i}}x^{\vec{i}})$ increases. The scale of $a_{\vec{i}}$ is dominated by the powers of p_1 at convergence. Theorem 6 and Lemma 2 further show that at convergence $1 - p_1$ also dominates a linear recurrence relation among the associated Taylor coefficients $a_{\vec{i}}$.

We use an example to explain this situation.

Example 2. Let $f = (1 - x_1 - x_2)(1 - 2x_1 - 3x_2)$, then

[3] Our discussion can be directly extended to the general case when $1 - p_1$ is the dominating factor in $(1 - p_1) \cdots (1 - p_t)$.

$1-(2x_1+3x_2) = 1-p_1$ is the dominating factor and

$$\frac{1}{f} = \frac{1}{(1-p_1)(1-p_2)} = \sum_{\vec{\imath}\in\mathbb{Z}^n_{\geq 0}} a_{\vec{\imath}} x^{\vec{\imath}}$$

$$= \frac{\overbrace{1-2x_1}^{r_1}}{1-\underbrace{(2x_1+3x_2)}_{p_1}} + \frac{\overbrace{-\frac{1}{3}+\frac{2x_1}{3}}^{r_2}}{1-\underbrace{(x_1+x_2)}_{p_2}}$$

$$+ \frac{\overbrace{\frac{1}{3}+\frac{5x_1}{3}-\frac{2x_1^2}{3}}^{r}}{\underbrace{(1-2x_1-3x_2)}_{1-p_1}\underbrace{(1-x_1-x_2)}_{1-p_2}}$$

$$= r_1\left(1+p_1+p_1^2+\cdots\right) + r_2\left(1+p_2+p_2^2+\cdots\right)$$
$$+ r\left(1+p_1+p_1^2+\cdots\right)\left(1+p_2+p_2^2+\cdots\right). \quad (20)$$

Now $A_i^{(1)} = p_1^i = (2x_1+3x_2)^i, A_i^{(2)} = p_2^i = (x_1+x_2)^i$. According to Theorem 5, as $\operatorname{multideg}(a_{\vec{\imath}} x^{\vec{\imath}})$ increases, each $a_{\vec{\imath}}$ can be collected from

$$S_{M,N} = r_1\left(A_M^{(1)}+\cdots+A_N^{(1)}\right) + r_2\left(A_M^{(2)}+\cdots A_N^{(2)}\right)$$
$$+ r\underbrace{\left(A_{M_1}^{(1)}A_{M_2}^{(2)}+\cdots+A_{N_1}^{(1)}A_{N_2}^{(2)}\right)}_{T_{M,N}}, \quad (21)$$

$M \leq M_1+M_2$, $N_1+N_2 \leq N$, and both M, $N\to\infty$. ⊠

We use $\chi_{a_{\vec{\imath}}}\left(r_1 A_j^{(1)}\right)$ to extract the contribution from $r_1 A_j^{(1)}$ to $a_{\vec{\imath}}$ in (20), meaning the coefficient of $x^{\vec{\imath}}$ in $r_1 A_j^{(1)}$. Since $1-p_1$ is the dominating factor, as $\operatorname{multideg}(x^{\vec{\imath}})$ increases, the contribution due to $r_1(A_M^{(1)}+\cdots+A_N^{(1)})$ dominates $a_{\vec{\imath}}$. In other words,

$$\lim_{M,N\to\infty} \frac{\chi_{a_{\vec{\imath}}}\left(r_2(A_M^{(2)}+\cdots+A_N^{(2)})\right)}{\chi_{a_{\vec{\imath}}}\left(r_1(A_M^{(1)}+\cdots+A_N^{(1)})\right)} = 0. \quad (22)$$

Combining Theorem 5 and (21), we write

$$a_{\vec{\imath}} = \chi_{a_{\vec{\imath}}}\left(S_{M,N}\right)$$
$$= \chi_{a_{\vec{\imath}}}\left(r_1(A_M^{(1)}+\cdots+A_N^{(1)})\right)$$
$$+ \chi_{a_{\vec{\imath}}}\left(r_2(A_M^{(2)}+\cdots+A_N^{(2)})\right)$$
$$+ \chi_{a_{\vec{\imath}}}\left(r\underbrace{(A_{M_1}^{(1)}A_{M_2}^{(2)}+\cdots+A_{N_1}^{(1)}A_{N_2}^{(2)})}_{T_{M,N}}\right) \quad (23)$$

for the associated M, N, as well as M_1, M_2 and N_1, N_2.

Theorem 6 is analogous to (7) that holds for a single irreducible factor.

THEOREM 6. *Under a monomial order, as* $\operatorname{multideg}(a_{\vec{\imath}} x^{\vec{\imath}})$ *increases,*

$$\lim_{M,N\to\infty} \frac{\chi_{a_{\vec{\imath}}}\left(S_{M+1,N+1}\right)}{\chi_{a_{\vec{\imath}}}\left(p_1\cdot S_{M,N}\right)} = 1. \quad (24)$$

PROOF. For a given $a_{\vec{\imath}}$, based on Theorem 5 we can always find M, N such that $a_{\vec{\imath}} = \chi_{a_{\vec{\imath}}}\left(S_{M,N}\right) = \chi_{a_{\vec{\imath}}}\left(S_{M+1,N+1}\right)$. Recall $p_1 = A_1^{(1)}$. We look at $p_1\cdot S_{M,N}$ and $S_{M+1,N+1}$:

$$p_1\cdot S_{M,N} = r_1\left(A_{M+1}^{(1)}+\cdots+A_{N+1}^{(1)}\right)$$
$$+ A_1^{(1)} r_2\left(A_M^{(2)}+\cdots+A_N^{(2)}\right)$$
$$+ r\underbrace{\left(A_{M_1+1}^{(1)}A_{M_2}^{(2)}+\cdots+A_{N_1+1}^{(1)}A_{N_2}^{(2)}\right)}_{A_1^{(1)}\cdot T_{M,N}},$$

$$S_{M+1,N+1} = r_1\left(A_{M+1}^{(1)}+\cdots+A_{N+1}^{(1)}\right)$$
$$+ r_2\left(A_{M+1}^{(2)}+\cdots+A_{N+1}^{(2)}\right)$$
$$+ r\left(A_1^{(1)}\cdot T_{M,N}+A_{M+1}^{(2)}+\cdots+A_{N+1}^{(2)}\right).$$

Then combine with the convergence property (22). □

LEMMA 2. *Under a monomial order, as* $\operatorname{multideg}(a_{\vec{\imath}} x^{\vec{\imath}})$ *increases,*

$$\lim_{\vec{\imath}\to\infty^n} \frac{a_{\vec{\imath}}}{b_1 a_{\vec{\imath}-\vec{d}_1}+b_2 a_{\vec{\imath}-\vec{d}_2}+\cdots+b_m a_{\vec{\imath}-\vec{d}_m}} = 1, \quad (25)$$

for the dominating factor $1-p_1$, *where* $p_1 = b_1 x^{\vec{d}_1}+b_2 x^{\vec{d}_2}+\cdots+b_m x^{\vec{d}_m}$.

PROOF. Expand p_1 and collect the coefficients corresponding to $a_{\vec{\imath}}$ in (24). □

Similar to (8), Lemma 2 provides a linear relation for the multivariate Taylor coefficients $a_{\vec{\imath}}$ at convergence. In other words, at convergence, the results in §2.2 can be applied to the dominating factor $1-p_1$. Based on such convergence properties, we build our method.

Algorithm: `DomFactor <floating point>`

Given a polynomial $f(x_1,\ldots,x_n)\in\mathbb{C}[x_1,\ldots,x_n]$ such that $f = 1-p = (1-p_1)\cdots(1-p_t)$ and that $1-p_1$ is the dominating factor. Compute $1-p_1$ in floating point arithmetic.

(1) [Expansion.] Obtain required Taylor coefficients $a_{\vec{\imath}}$ by expanding $1+p+p^2+\cdots$ for $f = 1-p$.

(2) [Rank of $\Delta_{\ell+1}^{(j)}$.] For a sufficiently large $j>0$, consider the rank of matrix $\Delta_{\ell+1}^{(j)}$ in (12) for $\ell = 0,1,2,\ldots$. The matrix $\Delta_{\ell+1}^{(j)}$ first becomes singular at $\ell = m$.[4]

(Within the constraint of finite precision, a larger j leads to a better convergent result. The choice of j depends on the desired accuracy and determines the required computational effort.)

(3) [Compute $1-p_1$.] Solve the corresponding linear system (9) for m obtained in Step (2). The solution of this system approximates the coefficients in the dominating factor $1-p_1 = 1-b_1 x^{\vec{d}_1}-\cdots-b_m x^{\vec{d}_m}$.

We comment on Step (2) of Algorithm `DomFactor`. Since f is given, an upper bound on $\operatorname{multideg}(p_1)$ is $\operatorname{multideg}(f) = \vec{d}_\lambda$. The step of determining m can be achieved by a single singular value decomposition procedure on matrix $\Delta_\lambda^{(j)}$.

[4]Now $1-p_1$ is assumed to have full terms, thus we have $\bar{\Delta}_{\ell+1}^{(v)} = \Delta_{\ell+1}^{(j)}$ for $\Delta_{\ell+1}^{(j)}$ defined in (10).

As a by-product, we present a semi-irreducibility test for a given multivariate polynomial in floating point arithmetic. We call it a semi-irreducibility test because a factorizable polynomial without a dominating factor can still pass the semi-irreducibility test.

Algorithm: `SemiIrredTest <floating point>`

Given a polynomial $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ such that $f = (1-p_1)\cdots(1-p_t)$ and there exists a dominating factor in f. Apparently, if $t = 1$ then f is irreducible. Determine whether f is irreducible in floating point arithmetic.

(1) [Expansion.] Obtain required Taylor coefficients $a_{\vec{\imath}}$ by expanding $1 + p + p^2 + \cdots$.

(2) [Rank of $\Delta_\lambda^{(j)}$.] Pick a sufficiently large $j > 0$. Check the rank of $\Delta_\lambda^{(j)}$ for multideg$(f) = \vec{d_\lambda}$.

(If matrix $\Delta_\lambda^{(j)}$ is non-singular and there exist a dominating factor in f, then f irreducible.)

So far we show that for a polynomial f having a dominating factor, we can determine whether f is irreducible and extract the dominating factor. However, in general a factorizable polynomial may not contain a dominating factor. This is the focus of our next subsection.

3.2 Partially Dominating Factors

We use an example to explain the behavior of partially dominating factors and show how to apply our results developed in §3.1 to this situation. As stated at the beginning of §3, all partially dominating factors are assumed to be full-termed.

Example 3. Consider $f = (1 - x_1 + x_2)(1 - 7x_1)$ and the expansion of its reciprocal

$$\frac{1}{f} = \frac{r_1}{1 - \underbrace{(x_1 - x_2)}_{p_1}} + \frac{r_2}{1 - \underbrace{7x_1}_{p_2}} + \frac{r}{\underbrace{(1 - x_1 + x_2)(1 - 7x_1)}_{(1-p_1)(1-p_2)}}$$
$$= r_1 (1 + p_1 + \cdots) + r_2 (1 + p_2 + \cdots) + r (1 + p_1 + \cdots)(1 + p_2 + \cdots) = \sum_{\vec{\imath} \in \mathbb{Z}^n_{\geq 0}} a_{\vec{\imath}} x^{\vec{\imath}}.$$

The powering of p_2 dominates the exponential growth in the coefficients of terms that favor the powering of x_1, but the contribution due to the powering of x_2 can only come from p_1. Unlike Example 1, there does not exist a dominating factor in f. Each of the two factors $1 - p_1$ and $1 - p_2$ dominates within a respective subset, K_1 or $\mathsf{K}_2 \subset \{a_{\vec{\imath}}\}_{\vec{\imath} \in \mathbb{Z}^n_{\geq 0}}$, as multideg$(a_{\vec{\imath}} x^{\vec{\imath}})$ increases. ⊠

Let $\mathsf{K}_1 \subset \mathbb{Z}^n_{\geq 0}$ be such that if multideg$(a_{\vec{\kappa}} x^{\vec{\kappa}})$ follows a monomial order and $\vec{\kappa} \in \mathsf{K}_1$, then the partially dominating factor $1 - p_1$ dominates $a_{\vec{\kappa}}$ as $\vec{\kappa}$ increases. Both Theorem 6 and Lemma 2 hold for $\{a_{\vec{\kappa}}\}_{\vec{\kappa} \in \mathsf{K}_1}$. To recover $1 - p_1$, Algorithm `DomFactor` can be carried out for $a_{\vec{\kappa}}$, with $\vec{\kappa}$ restricted in the subset K_1. Nevertheless, in general, such a subset is not given.

Recall that for recovering the dominating factor, Algorithm `DomFactor` is carried out for Taylor coefficients associated with a sufficiently large $j > 0$. To recover a set of partially dominating factors, we modify Algorithm `DomFactor` so that it is performed on a set of j's.

In order to recover all partially dominating factors, our strategy is to select a set of j's such that it intersects with each subset K_ℓ that is associated with a partially dominating factor. There are many ways to select such j's. For simplicity, in our discussion we fix an order, the graded reverse lex order. For reference, here we repeat the definition (see, e.g., [3, pp.58-59]).

Definition 1. *Let $\vec{d_i}, \vec{d_j} \in \mathbb{Z}^n_{\geq 0}$, $\vec{d_i} >_{\text{grevlex}} \vec{d_j}$ if*

$$\left|\vec{d_i}\right| = \sum_{k=1}^{n} d_{i_k} > \left|\vec{d_j}\right| = \sum_{k=1}^{n} d_{j_k}$$

or $|\vec{d_i}| = |\vec{d_j}|$ and the rightmost non-zero entry of $\vec{d_i} - \vec{d_j} \in \mathbb{Z}^n$ is negative.

Return to our polynomial f in Example 3. We use $J(\eta)$ to denote the collection of all $j > 0$ such that $|\vec{k_j}| = \eta$ in $a_{\vec{k_j}} x^{\vec{k_j}}$. If we pick a sufficiently large $\eta > 0$ and proceed with Algorithm `DomFactor` for each $j \in J(\eta)$, we encounter a set of matrices $\{\Delta_{\ell+1}^{(j)}\}_{j \in J(\eta)}$.

At $\ell = 2$, a subset of $\{\Delta_3^{(j)}\}_{j \in J(\eta)}$ are singular matrices, which reflect the partially dominating factor $1 - p_2$. Then at $\ell = 3$, there is another subset $\{\Delta_4^{(j)}\}_{j \in J(\eta)}$ of singular matrices reflecting the partially dominating factor $1-p_1$. For each of those singular matrices, an approximate polynomial factor $1 - \phi$ can be computed by solving a linear system (9). Figures 1 and 2 illustrate K_2 and K_1 in Example 3. Our computational environment is Maple 12 with Digits = 15.

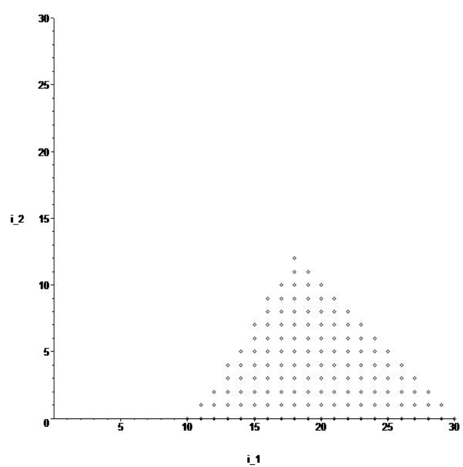

Figure 1: In Example 3, at $m = 2$, dotted (i_1, i_2) record $\|(1-\phi) - (1-p_2)\|_2 < 10^{-7}$ for the multivariate indices of Taylor coefficients such that $i_1 + i_2 = \eta = 0, \ldots, 30$.

Figures 1 and 2 illustrate the results for $\eta = 0, 1, \ldots, 30$. But for recovering factors $1-p_1$ and $1-p_2$, we only need to compute for a sufficiently large η, instead of several η such as in Figures 1 and 2 where $\eta = 0, 1, \ldots, 30$. We further remark that in practice, as long as we can determine both $1 - p_1$ and $1 - p_2$, for a fixed η we may not even need to compute $\Delta_{\ell+1}^{(j)}$ for all $j \in J(\eta)$.

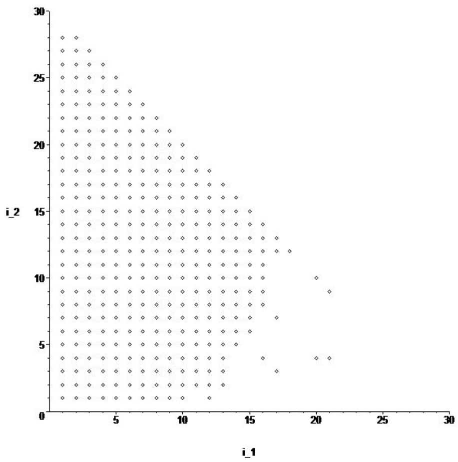

Figure 2: In Example 3, at $m = 3$, dotted (i_1, i_2) record $\|(1-\phi) - (1-p_1)\|_2 < 10^{-7}$ for the multivariate indices of Taylor coefficients such that $i_1 + i_2 = \eta = 0, \ldots, 30$.

Performing Algorithm `DomFactor` on a set $J(\eta)$ leads to an algorithm that computes a set of partially dominating factors.

Algorithm: `PartDomFactors <floating point>`
Given a polynomial $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ such that there does not exist a dominating factor in $f = 1 - b_1 x^{\vec{d}_1} - \cdots - b_m x^{\vec{d}_m} = (1-p_1) \cdots (1-p_t)$ and that $1-p_1, \ldots, 1-p_s$ are partially dominating factors. Compute $1-p_1, \ldots, 1-p_s$ in floating point arithmetic.

(1) [Expansion.] Obtain required Taylor coefficients $a_{\vec{\imath}}$ by expanding $1 + p + p^2 + \cdots$.

(2) [Rank of $\{\Delta_{\ell+1}^{(j)}\}_{j \in J(\eta)}$.] Pick a sufficiently large $\eta > 0$. For $\ell = 0, 1, \ldots, m$, consider the rank of matrix $\Delta_{\ell+1}^{(j)}$ for $j \in J(\eta)$.

Collect those $\Delta_{\ell+1}^{(j)}$ that first become singular (rank deficient by one) in a subset Q. At the end of this step, we have a subset Q that collects singular matrices $\Delta_{\ell+1}^{(j)}$ for $\ell = 0, 1, \ldots, m-1$.

(3) [Compute $1-p_1, \ldots, 1-p_s$.] Solve the corresponding linear system (9) for each matrix in Q obtained in the previous step. The solutions of these linear systems can be grouped to reflect partially dominating factors $1-p_1, \ldots, 1-p_s$ respectively.

3.3 Dealing with Non-Dominating Factors

We investigate Scenario 3 at the end of §2.3. Given a polynomial $f = (1-p_1) \cdots (1-p_t)$, if $1-p_1$ is a non-dominating factor, then there is at least another non-dominating factor, say, $1-p_2$. There does not exist a subsequence of the associated Taylor coefficients on which either factor contributes more than the powers of another.

Distinct non-dominating factors.

We use an example to explain our strategy to discover distinct non-dominating factors $1-p_1, \ldots, 1-p_s$ of f. That is, for $1 \le i \le j \le s$, whenever $i \ne j$, then $p_i \ne p_j$.

Example 4. Consider $f = (1 - x_1 - x_2)(1 - x_1 + x_2)$ and the expansion of its reciprocal

$$\frac{1}{f} = \frac{r_1}{1 - \underbrace{(x_1 + x_2)}_{p_1}} + \frac{r_2}{1 - \underbrace{(x_1 - x_2)}_{p_2}} + \frac{r}{\underbrace{(1 - x_1 - x_2)}_{1-p_1}\underbrace{(1 - x_1 + x_2)}_{1-p_2}} = \sum_{\vec{\imath}} a_{\vec{\imath}} x^{\vec{\imath}}.$$

Because p_1 and p_2 share the same magnitude of coefficients with respect to each variable, in any Taylor coefficient $a_{\vec{\imath}}$ the absolute values of the contributions from powering p_1 and p_2 are identical. Therefore none of them dominates in any subsequence of $\{a_{\vec{\imath}}\}_{\vec{\imath} \in \mathbb{Z}^n_{\ge 0}}$.

Suppose we proceed with Algorithm `PartDomFactors` on the associated set of Taylor coefficients $\{a_{\vec{\imath}}\}_{\vec{\imath} \in \mathbb{Z}^n_{\ge 0}}$. Since neither $1-p_1$ nor $1-p_2$ dominates, we reach singular matrices only at $\Delta_7^{(j)}$. But then the corresponding solution can only recover the product of $(1-p_1)(1-p_2)$ in its expanded form, instead of the individual factors, $1-p_1$ and $1-p_2$.

By shifting and rotating the representation basis, we can either turn these non-dominating factors into partially dominating factors, or one of them into a dominating factor, in the new representation basis.

Let $y_1 = -3 + x_1 + x_2$, $y_2 = 18 - 2x_1 - 4x_2$ and represent f in the basis of y_1, y_2, then both $1-p_1$ and $1-p_2$ become partially dominating factors in f,

$$\begin{aligned}\frac{1}{f} &= \frac{\theta_1}{1 - \underbrace{(3 + y_1)}_{p_1 \text{ in } y_1, y_2}} + \frac{\theta_2}{1 - \underbrace{(-9 + 3y_1 + y_2)}_{p_2 \text{ in } y_1, y_2}} \\ &\quad + \frac{\theta}{(1-p_1)(1-p_2)} \\ &= \frac{\bar{\theta}_1}{1 - (-0.5y_1)} + \frac{\bar{\theta}_2}{1 - (0.3y_1 + 0.1y_2)} \\ &\quad + \frac{\bar{\theta}}{(1-p_1)(1-p_2)}.\end{aligned}$$

If we represent f with respect to the basis of $z_1 = -3 + x_1 + x_2$ and $z_2 = 10 - 2x_1 - 4x_2$ then

$$\begin{aligned}\frac{1}{f} &= \frac{\vartheta_1}{1 - \underbrace{(3 + z_1)}_{p_1 \text{ in } z_1, z_2}} + \frac{\vartheta_2}{1 - \underbrace{(-1 + 3z_1 + z_2)}_{p_2 \text{ in } z_1, z_2}} \\ &\quad + \frac{\vartheta}{(1-p_1)(1-p_2)} \\ &= \frac{\bar{\vartheta}_1}{1 - (-0.5z_1)} + \frac{\bar{\vartheta}_2}{1 - (1.5z_1 + 0.5z_2)} \\ &\quad + \frac{\bar{\vartheta}}{(1-p_1)(1-p_2)}\end{aligned}$$

gives a representation in which $1-p_2$ becomes the dominating factor of f. ⊠

If there are non-dominating factors in f that are identical, then f has repeated factors. Such repeated factors remain identical in all bases and can not be separated by a change of basis.

Identical non-dominating factors.

Let $f = (1-p_1) \cdots (1-p_t)$. Suppose $1-p_1, \ldots, 1-p_s$ are non-dominating factors and $p_1 = \cdots = p_s$ for $1 \le s \le t$.

If we perform Algorithm `PartDomFactors` on the associated Taylor coefficients $\{a_{\vec{\imath}}\}_{\vec{\imath}\in\mathbb{Z}^n_{\geq 0}}$, we can only recover the product of $(1-p_1)\cdots(1-p_s) = (1-p_1)^s$ in its expanded form. Unlike the distinct non-dominating factors, these identical factors can not be separated by changing the representation basis.

Let $g = (1-p_1)^s$. By reformulating the factoring of g into another factorization problem, we can recover the power s and the irreducible factor $1-p_1$.

Dully substituting variable x_i by y_i in g, we obtain another multivariate polynomial $\bar{g} \in \mathbb{C}[y_1, \ldots, y_n]$ such that if $y_i = x_i$, then $\bar{g} = g$. Now we have

$$g = (1-p_1)^s, \qquad \bar{g} = (1-\bar{p}_1)^s,$$

in which $g, p_1 \in \mathbb{C}[x_1,\ldots,x_n]$ and $\bar{g}, \bar{p}_1 \in \mathbb{C}[y_1,\ldots,y_n]$.

Define another polynomial $F \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ such that

$$F = g - \bar{g} = (1-p_1)^s - (1-\bar{p}_1)^s. \tag{26}$$

If $s = 1$, then F is an irreducible polynomial in $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. This is because $\bar{p}_1 - p_1$ is irreducible in $(\mathbb{C}[y_1,\ldots,y_n])[x_1,\ldots,x_n]$.

If $s \geq 2$, then F can be factorized in $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$:

$$\begin{aligned} F &= g - \bar{g} = (1-p_1)^s - (1-\bar{p}_1)^s \\ &= \big((1-p_1) - (1-\bar{p}_1)\big)\left(\sum_{j=0}^{s-1}(1-p_1)^j(1-\bar{p}_1)^{s-1-j}\right) \\ &= (\bar{p}_1 - p_1)\left(\sum_{j=0}^{s-1}(1-p_1)^j(1-\bar{p}_1)^{s-1-j}\right). \end{aligned} \tag{27}$$

By considering the factorization of F over $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, we reformulate the problem of factorizing a multiple power of a repeated factor into the factorization of another square-free polynomial. If F is irreducible, we conclude that g is irreducible. If we can factorize F, then g is a multiple power of an irreducible polynomial $1-p_1$. From the factorization of F in (27), we can determine the irreducible factor $1-p_1$. Once both g and p_1 are known, the multiple power of s is determined by $\mathrm{multideg}(g) = s \cdot \mathrm{multideg}(p_1)$.

Algorithm: `PowerFactors <floating point>`

Given a polynomial $g(x_1,\ldots,x_n) \in \mathbb{C}[x_1,\ldots,x_n]$ in its expanded form such that $g = (1-p_1)^s$ and $1-p_1$ is irreducible. Determine whether $s = 1$; if not, determine s and $1-p_1$ in floating point arithmetic.

(1) [Factorize F.] Define $F \in \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ as in (26). Note that the constant of F is zero. In order to factorize F from the associated Taylor expansion, we need to introduce a constant term. So we form $\bar{F}$ by representing F in a shifted basis $\bar{x}_1 = x_1+\sigma_1, \ldots, \bar{x}_n = x_n+\sigma_n, \bar{y}_1 = y_1+\varsigma_1, \ldots, \bar{y}_n = y_n+\varsigma_n$. Perform Algorithm `PartDomFactors` to factorize $\bar{F}$.

(2) [Determine s and $1-p_1$.] If $\bar{F}$, hence F, cannot be factorized, then $s = 1$ and $g = 1-p_1$ is irreducible. Otherwise, from the factorization $1-p_1$ can be determined. Once $1-p_1$ is obtained, s is computed as $\mathrm{multideg}(g) = s \cdot \mathrm{multideg}(p_1)$.

4. COMPLETE FACTORIZATION AND IRREDUCIBILITY TESTING

Our method to extract one or more polynomial factors from a given polynomial f assumes that f is the numerical representation of either a factorizable or an irreducible polynomial. A different problem is that of so-called approximate factorization, where one returns the factorization of a factorizable polynomial that is closest to f in some sense, regardless whether f is numerically irreducible or not. For significant noise it remains to be answered whether our method can be adapted for this purpose.

As far as computational effort is concerned, our method depends on the size of the extracted factors rather than on the size of the input polynomial.

A multivariate polynomial is a product of a finite number of irreducible factors. As each irreducible factor can be extracted, our approach can lead to the complete factorization of multivariate polynomials and, as a by-product, an irreducibility test. Both of them belong to our current research and are commented below.

Toward complete factorization.

For a given multivariate polynomial $f = (1-p_1)\cdots(1-p_t)$, a subset of its irreducible factors can be extracted from the Taylor expansion of $1/f$. Such extracted factors, $1-p_1, \ldots, 1-p_s$, $1 \leq s \leq t$, can be the dominating factor, partially dominating factors, or non-dominating factors.

Let $h_1 = (1-p_1)\cdots(1-p_s)$. We continue to extract irreducible factors from $f_1 = f/h_1$. Since there are only finitely many irreducible factors in f, all irreducible factors can be extracted after repeating this procedure. An approximate polynomial division can be found in [15]. But perturbation can be introduced after each approximate polynomial division.

On the other hand, by shifting and rotating the basis in the polynomial representation, the dominating, partially dominating, or non-dominating factors to be extracted can change type. If each time different irreducible factors are extracted from the given polynomial f, then eventually all irreducible factors are extracted. Compared to polynomial division, changing the basis does not perturb the input polynomial. However, in order to achieve complete factorization, an effective strategy for changing basis is required such that the extraction of all irreducible factors can be guaranteed.

Toward an irreducibility test.

A factorizable polynomial without a dominating factor can pass Algorithm `SemiIrredTest`. By performing Algorithm `SemiIrredTest` on a set of j's that intersects with subsets associated with partially dominating factors, a polynomial with partially dominating factors can be captured.

Still, a factorizable polynomial with non-dominating factors can pass the above modified test. Following the discussion in §3.3, the changing of basis in the polynomial representation and Algorithm `PowerFactors` for factoring repeated factors provide us with tools to design an absolute irreducibility test for complex multivariate polynomials in floating point arithmetic.

Acknowledgements

We thank Markus Hitz, Erich Kaltofen, and an anonymous referee for their valuable remarks.

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