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On the probability of abandonment in queues with limited sojourn and waiting times

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Abstract

Consider the Geo/Geo/1 queue with impatient customers and let X reflect the patience distribution. We show that systems with a smaller patience distribution X in the convex-ordering sense give rise to fewer abandonments (due to impatience), irrespective of whether customers become patient when entering the service facility.

Keywords: Geo/Geo/1 queue; Impatient customers; Probability of abandonment; Convex-ordering

1. Introduction

The study of single-server queues with impatient customers has a long history. It seems Palm [8] was the first to consider customer impatience. Barrar [2] analyzed the M/M/1+D system, while a key reference for the general GI/GI/1+GI is Baccelli et al. [1]. In this work a stability condition was established for the general case, while for the M/GI/1+GI queue the virtual waiting time was studied. Markovian arrivals were considered by Combé [6], who studied the MAP/G/1+M queue and derived an expression for the transform of the virtual waiting time and the probability of abandonment. Most of these studies assume that a customer becomes patient when entering the server (that is, the system has a limited waiting time) and set up a Markov process using the virtual (offered) waiting time. Van Houdt et al. [10] developed an algorithm to compute the response time distribution in a D-MAP/PH/1+D queue, by setting up a finite GI/M/1-type Markov chain, which was generalized in [11] by allowing the patience distribution to be general, as opposed to deterministic. In these studies systems with both limited waiting and sojourn times were considered.

Bhattacharya and Ephremides [3] showed, for the GI/GI/m+GI queue, that the number of customers abandoning the system over any time interval decreases stochastically when the patience distribution becomes stochastically larger (i.e., when the patience distribution X_1 is replaced by X_2 with $P[X_2 \geqslant x] \geqslant P[X_1 \geqslant x]$, for all $x \geqslant 0$). Thus, as intuitively expected, more patience leads to fewer

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abandonments. An interesting open question therefore is: What can be said about the probability of abandonment for different patience distributions X with the same mean, finite amount of patience $m=E[X]$? Making use of the results by Brandt and Brandt [5, Theorem 3.1] or Boxma and de Waal [4] it can be shown easily that in a M/M/1+GI queue, where customers become patient when entering the service facility, smaller distributions (in the convex-ordering sense) give rise to fewer abandonments. In both these papers an expression for the probability of abandonment is established via the steady-state probabilities of the virtual offered waiting time. To the best of our knowledge this is the only result of this type available in the literature.

In this paper we prove the discrete-time counterpart of the M/M/1+GI result mentioned above as well as the more difficult system where customers remain impatient after entering the service facility, that is, the Geo/Geo/1+GI queue with a limited waiting or sojourn time. In the latter case customers are assumed to be unaware of the required sojourn time upon arrival and may therefore receive partial service (and waste some of the service capacity). An expression for the probability of abandonment is derived from the age process as opposed to the virtual offered waiting time. Furthermore, we demonstrate that the number of abandonments in a single-server queue with impatient customers and geometric service times, i.e., the ·/Geo/1+GI queue, is higher for systems with a limited sojourn time as opposed to a limited waiting time.

Let X be a general, discrete patience distribution on the non-negative integers and assume without loss of generality that $P[X=0]=0$. Further assume that for each X there exists some $r\geqslant 0$ sufficiently large, such that $P[X>r]=0$, i.e., the maximum amount of patience that a customer can have is bounded above by some constant r (note that we do not assume that a single r exists for all distributions X, but given a particular X such an r can be found). The results in this paper can be generalized to include distributions for which such an r does not exist ($P[X=\infty]=0$ for such distributions as $m=E[X]$ is finite, implying that the Geo/Geo/1+X queue is stable [1]). In Sections 2 and 3 we show that the smaller distributions (in the convex-ordering sense) induce a lower probability of abandonment in a Geo/Geo/1+GI system with a limited sojourn and waiting time, respectively. The lower number of abandonments achieved by ·/Geo/1+GI systems with limited waiting times, as opposed to limited sojourn times, is proven in Section 4.

2. The Geo/Geo/1 queue with a limited sojourn time

In this section, we consider a Geo/Geo/1 queue with $P[arrival]=\alpha$, $P[departure]=\beta$ and with impatient customers. Customers are assumed to be impatient irrespective of whether they are being served or not. Let X represent the patience distribution of the customers and denote $P[X=k]$ as $a_k(X)$ and $P[X\leqslant k]$ as $p_k(X)$. To simplify the notation, we shall write

$$x_k(X)=(1-p_k(X))\alpha,$$

$$y_k(X)=\beta+\frac{a_k(X)}{1-p_{k-1}(X)}(1-\beta) =\frac{a_k(X)+(1-p_k(X))\beta}{1-p_{k-1}(X)},$$

for $k=0,\ldots,r$. When there is no ambiguity as to which distribution X is meant, we will drop the X to simplify the notation. Similarly, we will add an X (or Y) to any other variable when there is a need to clarify the patience distribution at hand. Note, x_k is the arrival rate (probability) of customers whose patience is more than k, while y_k is the probability that a customer with age k leaves the server (due to either impatience or a service completion) given that an age k customer occupied the server. Let A_n be the age of the customer in service just prior to time n, where A_n is said to be zero if the system is idle. All events, such as service completions, arrivals, etc. are assumed to occur immediately after time n, implying amongst others that the age of a customer in the service facility is at least one. As discussed below, $(A_n)_{n\geqslant 0}$ is a Markov chain, the $(r+1)\times(r+1)$ transition matrix of which is given by

$$P=\begin{bmatrix} b_1 & b_0 & 0 & 0 & \ldots & 0 & 0 \\ b_2 & a_1^1 & a_0^1 & 0 & \ldots & 0 & 0 \\ b_3 & a_2^2 & a_1^2 & a_0^2 & \ddots & 0 & 0 \\ b_4 & a_3^3 & a_2^3 & a_1^3 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ b_r & a_{r-1}^{r-1} & a_{r-2}^{r-1} & a_{r-3}^{r-1} & \ddots & a_1^{r-1} & a_0^{r-1} \\ b_{r+1} & a_r^r & a_{r-1}^r & a_{r-2}^r & \ldots & a_2^r & a_1^r \end{bmatrix}$$

with

$$
\begin{aligned}
&b_0 = x_0,\\
&b_1 = 1 - x_0,\\
&b_{i+1} = y_i \prod_{k=1}^{i} (1 - x_{i-k}), \quad \text{for } 1 < i \leqslant r,\\
&a_0^i = 1 - y_i, \ \text{for } 1 \leqslant i < r,\\
&a_{l+1}^i = y_i x_{i-l-1} \prod_{k=1}^{l} (1 - x_{i-k}),\\
&\quad \text{for } 0 < i \leqslant r \text{ and } 0 \leqslant l < i.
\end{aligned} \tag{1}
$$

Throughout the paper we assume that $(\prod_{k=i}^{j} \ldots) = 1$ for $i > j$. The expressions for b_0, b_1 and a_0^i follow in a straightforward manner from the definitions of x_k and y_k. Denote n as the current time instant and assume an age i customer is in service, that is, $A_n = i$, then a_{l+1}^i equals the probability that this customer leaves the system (which happens with a probability y_i, notice that $y_r = 1$ since there can never be a customer with an age larger than r in the queue) and there will be a customer of age $i-l$ in service at time $n+1$. Therefore, every customer who arrived at some time $n-i+k$, with $0 < k \leqslant l$, should reach his critical age no later than time n (this occurs with probability $\prod_{k=1}^{l}(1 - x_{i-k})$), and finally, with probability x_{i-l-1} there is an arrival at time $n - i + l + 1$ of a customer whose patience is at least $i - l$ time units. A similar argument can be given with relation to the expression for b_{i+1}, the only difference being that b_{i+1} corresponds to a transition to an idle queue and hence no customer will be served at time instant $n + 1$.

Lemma 1.

$$
\prod_{k=k_{\min}}^{k_{\max}} (1 - z_k) = 1 - \sum_{k=k_{\min}}^{k_{\max}} z_k \prod_{m=k_{\min}}^{k-1} (1 - z_m),
$$

for $1 \leqslant k_{\min} \leqslant k_{\max}$.

Proof. The statement follows by induction on $k_{\max}$. □

Theorem 1. *The steady-state probabilities of the Markov chain* $(A_n)_{n \geqslant 0}$ *under consideration are given by*

$$
\begin{aligned}
\pi_0 &= \frac{1}{N},\\
\pi_i &= \frac{1}{N} \left(\prod_{k=1}^{i-1} \frac{1 - y_k}{1 - x_{k-1}} \right) \frac{x_0}{1 - x_{i-1}},
\end{aligned} \tag{2}
$$

for $i = 1, \ldots, r$ *and with* $N = 1 + \sum_{i=1}^{r} \left(\prod_{k=1}^{i-1} [(1 - y_k)/(1 - x_{k-1})] \right) x_0/(1 - x_{i-1})$.

Proof. Clearly, $\pi = (\pi_0, \pi_1, \ldots, \pi_r)$ is a stochastic vector, therefore, it suffices to verify whether π is an invariant vector of P. By making use of Eqs. (1) and (2) in the first balance equation $\sum_{i=0}^{r} \pi_i b_{i+1} = \pi_0$ and by eliminating the common factors in the numerator and denominator we have

$$
\begin{aligned}
&(1 - x_0) + \sum_{i=1}^{r} x_0 y_i \prod_{k=1}^{i-1} (1 - y_k)\\
&\quad = 1 \Leftrightarrow \sum_{i=1}^{r} y_i \prod_{k=1}^{i-1} (1 - y_k) = 1,
\end{aligned}
$$

which is proven by applying Lemma 1 with $k_{\min} = 1$, $k_{\max} = r$ and $z_k = y_k$, because $y_r = 1$. Inserting the expressions for b_0, a_i^i and π_i into the second balance equation $\pi_0 b_0 + \sum_{i=1}^{r} \pi_i a_i^i = \pi_1$ allows us to rewrite this as

$$
\begin{aligned}
&x_0 + \sum_{i=1}^{r} \frac{x_0^2 y_i}{1 - x_0} \prod_{k=1}^{i-1} (1 - y_k)\\
&\quad = \frac{x_0}{1 - x_0} \Leftrightarrow \sum_{i=1}^{r} y_i \prod_{k=1}^{i-1} (1 - y_k) = 1.
\end{aligned}
$$

Finally, the $(s + 1)$th balance equation $\sum_{i=s-1}^{r} \pi_i a_{i-s+1}^i = \pi_s$, for $1 < s \leqslant r$, is given by

$$
\begin{aligned}
&\left(\prod_{k=1}^{s-1} \frac{1 - y_k}{1 - x_{k-1}} \right) x_0\\
&\quad + \sum_{i=s}^{r} \left(\frac{x_0 x_{s-1} y_i \prod_{k=1}^{i-1} (1 - y_k)}{\prod_{k=1}^{s} (1 - x_{k-1})} \right) = \pi_s N.
\end{aligned}
$$

By the definition of π_s we have

$$(1-x_{s-1})\pi_s N + \frac{x_0 x_{s-1}\prod_{k=1}^{s-1}(1-y_k)}{\prod_{k=1}^{s}(1-x_{k-1})} \times \left(\sum_{i=s}^{r} y_i \prod_{k=s}^{i-1}(1-y_k)\right) = \pi_s N$$
$$\Leftrightarrow (1-x_{s-1})\pi_s + x_{s-1}\pi_s = \pi_s,$$

due to Lemma 1 with $k_{\min}=s$, $k_{\max}=r$ and $z_k=y_k$, as $y_r=1$.

Having determined the steady-state probabilities of the Markov chain $(A_n)_{n\geqslant 0}$, we can easily find an expression for the rejection probability P_I that a customer leaves the system before his service is completed/started: $P_I = 1-\sum_{i=1}^{r} P[Z=i]$. Here, $P[Z=i]$ is the probability that a customer is completely served and has a response time (i.e., waiting time + service time) equal to i time units. This probability is given by $P[Z=i]=(\beta/\alpha)\pi_i$, as $\beta\pi_i$ is the probability that at an arbitrary time epoch, an age i customer leaves the system after completing service, while α is the probability that an arrival occurs in an arbitrary slot. Using the stochastic nature of the vector π, it follows that $P_I = 1-(\beta/\alpha)(1-\pi_0)$.

Keeping α and β fixed, P_I is a function of the patience distribution X only; as such we refer to it as $P_I(X)$. We will prove that $P_I(X)\leqslant P_I(Y)$ if X is smaller than Y in the convex-ordering sense, i.e., $X\leqslant_{\rm cx} Y$ [9]. As a consequence, given a mean m, $P_I(X)$ is minimal for the deterministic distribution with mean m among all discrete distributions X on $\{1,2,\ldots\}$ with $E[X]=m$ (provided that m is an integer, otherwise X with $P[X=\lfloor m\rfloor]=\lceil m\rceil - m$ and $P[X=\lceil m\rceil]=m-\lfloor m\rfloor$ realizes the lowest $P_I(X)$). We start with the following lemma.

Lemma 2. *If* $X\leqslant_{cx} Y$, *then*

$$\prod_{k=0}^{i-1}(1-x_k(X)) \leqslant \prod_{k=0}^{i-1}(1-x_k(Y)).$$

Proof. Having $X\leqslant_{\rm cx} Y$ means that $\sum_{k=0}^{i-1} x_k(X) \geqslant \sum_{k=0}^{i-1} x_k(Y)$, for all $i\geqslant 1$ [9]. Moreover, $x_k(X)$ and $x_k(Y)$, $k\geqslant 0$, are both non-increasing rows. Thus, the vector $(x_0(X),\ldots,x_{i-1}(X))$ weakly majorizes $(x_0(Y),\ldots,x_{i-1}(Y))$ in the sense of Marshall and Olkin [7]. A theorem by Tomić [7, Proposition 4.B.2] implies that $\sum_{k=0}^{i-1} g(x_k(X)) \geqslant \sum_{k=0}^{i-1} g(x_k(Y))$ for any continuous increasing convex function g. Letting $g(x)=-\log(1-x)$ proves the lemma. □

Theorem 2. *Let* $X\leqslant_{\rm cx} Y$, *then* $P_I(X)\leqslant P_I(Y)$, *meaning a smaller patience distribution* (*in the convex-ordering sense*) *achieves a lower probability of abandonment in a* Geo/Geo/1 *queue with impatient customers in the system.*

Proof. Based on Theorem 1 and the expression found for P_I, it is sufficient to show $\Theta(X)\geqslant\Theta(Y)$, where Θ is defined as

$$\Theta(X) = \sum_{i=1}^{r(X)} \left(\prod_{k=1}^{i-1} \frac{1-y_k(X)}{1-x_{k-1}(X)}\right) \frac{x_0(X)}{1-x_{i-1}(X)}$$
$$= \sum_{i=1}^{r(X)} \frac{(1-p_{i-1}(X))(1-\beta)^{i-1} x_0(X)}{\prod_{k=1}^{i}(1-x_{k-1}(X))}$$
$$= \sum_{i=1}^{r(X)} \frac{x_{i-1}(X)(1-\beta)^{i-1}}{\prod_{k=0}^{i-1}(1-x_k(X))},$$

where the first equality is obtained by rewriting $(1-y_k(X))$ as $(1-\beta)(1-p_k(X))/(1-p_{k-1}(X))$ and the second by observing that $x_0(X)=\alpha$. Having $X\leqslant_{cx} Y$ yields $r(X)\leqslant r(Y)$. Rewrite $\Theta(X)\leqslant\Theta(Y)$ by subtracting the first $r(X)$ terms of $\Theta(Y)$ on both sides of the inequality and by dividing them by $(1-\beta)^{r(X)}$. As $0<\beta<1$, it now suffices to show

$$\sum_{i=1}^{r(X)} \frac{x_{i-1}(X)}{\prod_{k=0}^{i-1}(1-x_k(X))} \geqslant \sum_{i=1}^{r(Y)} \frac{x_{i-1}(Y)}{\prod_{k=0}^{i-1}(1-x_k(Y))}. \quad (3)$$

Dividing both sides of the equality given in Lemma 1 by $\prod_{m=k_{\min}}^{k_{\max}}(1-z_m)$ results in

$$\sum_{k=k_{\min}}^{k_{\max}} \frac{z_k}{\prod_{m=k}^{k_{\max}}(1-z_m)} = \frac{1}{\prod_{k=k_{\min}}^{k_{\max}}(1-z_k)} - 1.$$

By combining this equation for $k_{\min}=1$, $k_{\max}=r(X)$ (and $r(Y)$), and $z_k=x_{r(X)-k}$ (and $x_{r(Y)-k}$), with (3) we find

$$\frac{1}{\prod_{k=0}^{r(X)-1}(1-x_k(X))} \geqslant \frac{1}{\prod_{k=0}^{r(Y)-1}(1-x_k(Y))},$$

and this inequality is valid because of Lemma 2 (notice $1-x_k(X)=1$, for $k\geqslant r(X)$). □

In the next section we prove that this theorem is also valid for the Geo/Geo/1 queue where impatient customers become patient when entering the service facility.

Remark 1. The formula $P_I = 1-(\beta/\alpha)(1-\pi_0)$, which we can be rewritten as $\alpha(1-P_I)/\beta = 1-\pi_0$, is somewhat unexpected as one might get the incorrect impression that only successful customers occupy the service facility. However, the mean service time of successful customers (arriving with a rate $\alpha(1-P_I)$) is less than $1/\beta$, while the remaining service capacity is wasted by customers leaving the system prematurely.

3. The Geo/Geo/1 queue with a limited waiting time

Consider the same Geo/Geo/1 queue as in Section 2, but instead of having a limited sojourn time we assume that the limitation applies to the waiting time. This implies that once a customer has entered the service facility, he remains there until his service is completed. We further assume that a customer abandons the system even if he reaches his critical age at the exact time instant that the server becomes available to him. Let $\hat{A}_n$ be the age of the customer in service just prior to time n, where $\hat{A}_n$ is said to be zero if the system is idle. Notice, due to the patient nature of the customers in the service facility and the unbounded geometric service time, the state space of the process $(\hat{A}_n)_{n\geqslant 0}$ is infinite and equals $\{0,1,2,\ldots\}$. It is easily seen that $(\hat{A}_n)_{n\geqslant 0}$ is a Markov chain. Moreover, if we censor this Markov chain on the state space $\{0,1,2,\ldots,r\}$ its transition matrix $\hat{P}_r$ is identical to P if, for $i=1,\ldots,r-1$, we replace y_i by β in the various expressions of Eq. (1) (recall, $y_r=1$). Indeed, the probability that an age k customer leaves the service facility equals β as opposed to y_i as in Section 2 and the probability that the Markov chain $(\hat{A}_n)_{n\geqslant 0}$ makes a transition from some state $r+k$, for $k>0$, to a state $i \in \{0,\ldots,r\}$ does not depend upon k, meaning we can act as if age r customers simply leave the system with probability 1 when censoring on $\{0,1,2,\ldots,r\}$.

Theorem 3. *The steady-state probabilities of the Markov chain $(\hat{A}_n)_{n\geqslant 0}$ under consideration are given by*

$$\hat{\pi}_0 = \frac{1}{\hat{N}},$$
$$\hat{\pi}_i = \frac{1}{\hat{N}} \frac{(1-\beta)^{i-1}x_0}{\prod_{k=1}^{i}(1-x_{k-1})},$$

for $i\geqslant 0$ and with $\hat{N} = 1+\sum_{i=1}^{\infty}\left(\frac{(1-\beta)^{i-1}x_0}{\prod_{k=1}^{i}(1-x_{k-1})}\right)$.

Proof. For $0\leqslant i\leqslant r$, define $\pi_i(\beta,y)$ as π_i with y_k, for $1\leqslant k\leqslant r-1$, replaced by β (notice, $y_r=1$ is not to be replaced by β). Due to the censoring argument presented above we have that $(\hat{\pi}_0,\ldots,\hat{\pi}_r)$ is proportional to $(\pi_0(\beta,y),\ldots,\pi_r(\beta,y))$. Clearly, $\hat{\pi}_i = \hat{\pi}_r(1-\beta)^{i-r}$, for $i\geqslant r$. Hence, for $i\geqslant r$, $\hat{\pi}_i$ is proportional to $\pi_r(\beta,y)(1-\beta)^{i-r} = \pi_r(\beta,y)(1-\beta)^{i-r}/(\prod_{k=r+1}^{i}(1-x_k))$ as $x_k=0$ for $k>r$. Using the expression for π_i in Eq. (2), we find that $\hat{\pi}_i$, for $i\geqslant 0$, is proportional to $(1-\beta)^{i-1}x_0/(\prod_{k=1}^{i}(1-x_{k-1}))$, which proves the theorem as $(\hat{\pi}_0,\hat{\pi}_1,\hat{\pi}_2,\ldots)$ is a stochastic vector. □

For α and β fixed, the probability of abandonment is a function of the patience distribution X only, therefore, we refer to it as $\hat{P}_I(X)$.

Theorem 4. *Let $X\leqslant_{cx} Y$; then $\hat{P}_I(X)\leqslant\hat{P}_I(Y)$, meaning a smaller patience distribution (in the convex-ordering sense) achieves a lower probability of abandonment in a* Geo/Geo/1 *queue with impatient customers in the waiting room.*

Proof. Analogous to Section 2, we can establish the following relation $\hat{P}_I(X) = 1-(\beta/\alpha)(1-\hat{\pi}_0(X))$. Hence, $\hat{P}_I(X)\leqslant\hat{P}_I(Y)$ if $\Phi(X)\geqslant\Phi(Y)$, which we define as:

$$\Phi(X) = \sum_{i=1}^{\infty}\left(\frac{(1-\beta)^{i-1}}{\prod_{k=1}^{i}(1-x_{k-1}(X))}\right).$$

This inequality holds due to Lemma 2. □

Remark 2. Theorem 4 also applies for the Geo/Geo/1 queue with waiting time-aware customers as the loss probability in a GI/GI/1+GI queue is not affected by the awareness [1]. Waiting time-aware customers only enter the waiting room provided that they can be served without losing patience.

4. Queues with impatient customers and geometric service times

Consider a discrete-time first-come-first-served (FCFS) queueing system S with a single server that serves customers in a geometric time. Label the arriving, impatient customers in order of their arrival starting from 0 and let X be their patience distribution. We shall refer to the queueing system S in which customers remain impatient during their sojourn time as *SS* and the one where customers are only impatient while waiting as *SW*.

Lemma 3. *Consider an arbitrary realization* ω *of* S; *then if customer i is being served in the SS system at time n, there will be a customer j present in the service facility of the SW system at time n, with* $j \leqslant i$.

Proof. The result is immediate from a sample-path argument. □

Theorem 5. *The loss probability* $P_I(SW) \leqslant P_I(SS)$ *for any patience distribution X, meaning the* ·/Geo/1 *queue where customers are only impatient while waiting* (*as opposed to during their sojourn time*) *achieves fewer abandonments*.

Proof. Lemma 3 shows us that if the *SS* system is busy, then so is the *SW* system, i.e., $P_{\text{idle}}(SW) \leqslant P_{\text{idle}}(SS)$. In [11] it is shown that the relation $P_I = 1 - \frac{\beta}{\lambda}(1 - P_{\text{idle}})$ holds both in the *SS* and *SW* system, where λ is the stationary arrival rate, yielding $P_I(SW) \leqslant P_I(SS)$. □

Remark 3. Theorem 5 does not generally apply to any queue with impatient customers (see [10] for an example).

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