RAMSEY THEORY

1 Ramsey Numbers

Party Problem: Find the minimum number $R(k,l)$ of guests that must be invited so that at least k will know each other or at least l will not know each other (we assume that if A knows B, then B knows A as well).

Let us rephrase this problem in graph theoretical terms:

DEFINITION 1.1: A complete graph G is a graph in which each pair of vertices is connected by one edge (no loops). We denote the complete graph with n vertices as K_n.

DEFINITION 1.2: The Ramsey Number $R(k,l)$ is defined as the minimum number N such that for any coloring c of the set of edges of K_N, denoted as $E(K_N)$, K_N contains a *red* K_k or a *blue* K_l as a subgraph. A coloring c is a function from $\{(i,j)|i \neq j \text{ and } i,j \in \{1,\ldots,N\}\}$ to $\{red, blue\}$.

Some obvious properties are: $R(s,t) = R(t,s)$ and $R(s,2) = s$.

THEOREM 1.1 (Ramsey 1930): $R(s,t)$ is finite for all $s,t \geq 2$ and for $s,t > 2$ we have $R(s,t) \leq R(s-1,t) + R(s,t-1)$.

Proof: Select an arbitrary vertex v of the graph K_N, where $N = R(s-1,t) + R(s,t-1)$. Let c be an arbitrary coloring of K_N. Then, $R(s-1,t) + R(s,t-1) - 1$ edges arrive in v. Either $R(s-1,t)$ of them are *red* or $R(s,t-1)$ are *blue*. Without loss of generality, assume we have $R(s-1,t)$ vertices incident to v by means of red edges. These vertices form a $K_{R(s-1,t)}$ graph. Thus, for each coloring, including coloring c, we either have a blue K_t or a red K_{s-1} in this $K_{R(s-1,t)}$ graph. This completes the proof, as in the latter case a red K_s is formed by adding v to the red K_{s-1}. Q.E.D.

THEOREM 1.2: For all $s,t \geq 2$ we have $R(s,t) \leq \binom{s+t-2}{s-1}$.

Proof: Trivial for s or t equal to 2. For $s, t > 2$ (with induction on $s+t$), we use Ramsey's theorem and the fact that $\binom{k}{l} + \binom{k}{l-1} = (\frac{k-l+1}{l} + 1)\binom{k}{l-1} = \frac{k+1}{l}\binom{k}{l-1} = \binom{k+1}{l}$ (with $k = s+t-3$ and $l = s-1$).
Q.E.D.

Exercises 1.1: Prove the following identities:

1. $R(3,3) = 6$.
2. $R(3,4) > 8$.
3. $R(3,4) <= 9$; hence, $R(3,4) = 9$. [HINT: Consider the following three scenarios (i) at least 4 *red* edges arrive in some vertex v, (ii) at least 6 *blue* edges arrive in some vertex v and (iii) exactly 3 *red* and 5 *blue* edges arrive in all vertices v.]
4. $R(s,t) \leq R(s-1,t) + R(s,t-1) - 1$ if both $R(s,t-1)$ and $R(s-1,t)$ are even.
5. $R(s,s) \leq 2^{2s-3}$ [HINT: Let c be an arbitrary coloring of $K_{2^{2s-3}}$. Select an arbitrary vertex v_1, then there exists a set V_1 with at least 2^{2s-4} vertices such that $c(v_1v) = c(v_1w)$ for all $v, w \in V_1$. Let v_i be any vertex in V_{i-1}, let $V_i \subset V_{i-1}$ be a set with at least 2^{2s-3-i} vertices for which $c(v_iv) = c(v_iw)$ for all $v, w \in V_i$. Repeat this argument for $i = 2, \ldots, 2s-3$.]
6. $R(3,5) = 14$.

Ramsey numbers are very hard to compute, so far only the following are known: $R(2,t) = t$, $R(3,3) = 6$, $R(3,4) = 9$, $R(3,5) = 14$, $R(3,6) = 18$, $R(3,7) = 23$, $R(3,8) = 28$, $R(3,9) = 36$, $R(4,4) = 18$ and $R(4,5) = 25$. No other Ramsey numbers are currently known (upper and lower bounds exist).

In order to make a link with other mathematical disciplines we need to introduce the following numbers:

Definition 1.3: The generalized Ramsey numbers $R^{(q)}(a_1, a_2, \ldots, a_k)$ are defined as the minimum number N such that no matter how each q-element subset of an N-element set is colored with k colors, there exists an $i \in \{1, \ldots, k\}$ such that there is a subset of size a_i, all of whose q-element subsets have color i. [Remark: $R(k,l) = R^{(2)}(k,l)$]

THEOREM 1.3 (Ramsey 1930): All generalized Ramsey numbers are finite.

EXERCISES 1.2: On generalized Ramsey numbers:

1. Simplify $R^{(r)}(r, a_2, \ldots, a_k)$.
2. Express the Pigeon Hole Principle by means of a Ramsey number [Recall: Distributing $(n-1)t+1$ balls in t urns results in at least one urn with n balls].
3. Prove the Erdos-Szekeres Theorem (1935) using the $R^{(1)}(., \ldots, .)$ numbers [Theorem: any row of $ab+1$ distinct real numbers contains either an increasing subrow for size $a+1$ or a decreasing subrow of size $b+1$].
4. Prove the Schur Theorem (1916) using the $R^{(2)}(., \ldots, .)$ numbers [Theorem: for any natural number t, there exists an N sufficiently large such that for any partitioning $A_1, \ldots, A_t$ of $\{1, \ldots, N\}$ there exists an $i \in \{1, \ldots, t\}$ and and x, y and z in A_i such that $x + y = z$].
5. Prove the Erdos-Szekeres Theorem (1935) using the $R^{(4)}(., .)$ numbers [Theorem: for any n there exists an N finite such that from any N points in the plane (no 3 are collinear) some n are in a convex position. A set of n points in the plane is convex if any triangle formed by 3 of these n points does not contain another of the $n-3$ points].
6. $R^{(2)}(3, 3, 3) \leq 17$.
7. Prove the following identity: $R^{(r)}(a_1, \ldots, a_k) \leq R^{(r-1)}(R^{(r)}(a_1 - 1, a_2, \ldots, a_k), R^{(r)}(a_1, a_2-1, \ldots, a_k), \ldots, R^{(r)}(a_1, a_2, \ldots, a_k-1))+1$ for $a_1, a_2, \ldots, a_k > r$.

THEOREM 1.4 (Ramsey 1930): Let r and k be natural numbers, let A be an infinite set and let c be a k-coloring of $A^{(r)}$, then A contains a monochromatic infinite set[1].

[1] $X^{(r)}$ denotes the set of all r-subsets in X.

Proof (*): The theorem is trivial for $r = 1$. Hence, we prove the theorem by induction on r. Let c be an arbitrary k-coloring of A. We start by defining an infinite subset $\{x_1, x_2, \ldots\}$ of A and a nested sequence $B_0 \supset B_1 \supset \ldots$ of infinite subsets of A. Let $B_0 = A$, then B_l and x_l are constructed as follows. Pick $x_l \in B_{l-1}$ arbitrary and set $C_{l-1} = B_{l-1} - \{x_l\}$. Next, define $\tilde{c}$ as a coloring on $C_{l-1}^{(r-1)}$ by putting $\tilde{c}(\sigma) = c(\sigma \cup \{x_l\})$, where σ is an $r-1$-subset of C_{l-1}. By induction, C_{l-1} contains an infinite monochromatic set, say with color c_i, which we define B_l. Notice, for any $r-1$-subset σ in B_l we have $c(\sigma \cup x_l) = c_i$. Finally, define $c'(x_l) = c_i$.
Having constructed the infinite set X, it is clear that an infinite subset X_j of X exists, such that for some color c_i, we have $c'(x) = c_i$ for all $x \in X_j$. Then, each r-subset $\{x_{i_1}, \ldots, x_{i_r}\}$ of X_j has $c(\{x_{i_1}, \ldots, x_{i_r}\}) = c_i$. Indeed, let $i_{min} = \min_{n=1}^{r} i_n$, then $c(\{x_{i_1}, \ldots, x_{i_r}\}) = c'(x_{i_{min}}) = c_i$. (because $x_{i_n} \in B_{i_{min}}$ for $i_n > i_{min}$)

Q.E.D.

Exercises 1.3: Prove the following statement:

1. An infinite row of real numbers contains either an infinite decreasing subrow or an infinite increasing subrow.

2 Hales-Jewett Numbers

Definition 2.1: A (combinatorial) hypercube (or grid) of dimension n and width l is defined as the set of all strings of length n using the letters of an alphabet $L = \{a, b, \ldots\}$ with l letters. We denote this set of strings as $W_n(L)$.

A 1-parameter word M is defined as a string where 1 or more letters are replaced by a parameter X, e.g., $cabbXcXaaXb$. Such a 1-parameter word represents all the strings that can be obtained by replacing X by a letter in L, e.g., $\{cabbacaaaab, cabbbcbaabb, cabbcccaacb\}$. A 1-parameter word is sometimes referred to as a combinatorial line (in a hypercube). Similarly, we define a d-parameter word as a string where at least d letters are replaced by the parameters $X_1, \ldots, X_d$ and each parameter has to appear in the string, e.g., $caX_2ccX_1bbX_1X_1a$ is a 2-parameter word. A d-parameter word is often referred to as a d-dimensional subspace (of a hypercube) and reflects the l^d strings that can be obtained by replacing each parameter by a letter in L.

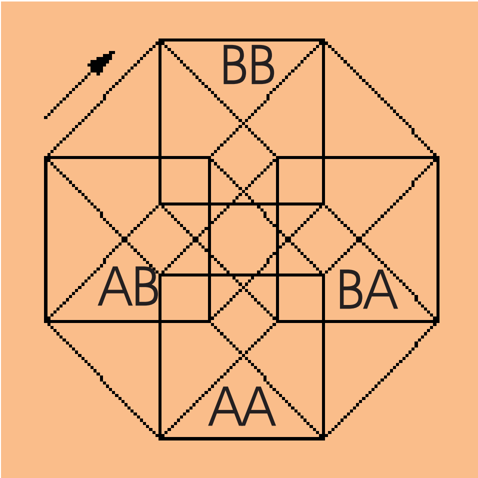

Fig. 1: Hypercube with dimension $n = 4$ and width $l = 2$

DEFINITION 2.2: The Hales-Jewett number $HJ(l,d,k)$ is defined as the smallest natural number such that for every k-coloring of an $HJ(l,d,k)$-dimensional hypercube with width l, there exists a monochromatic subspace of dimension d.

Clearly, $HJ(l,d,1) = d$ and $HJ(1,d,k) = d$. In order to prove the finiteness of the Hales-Jewett numbers we start with the following two lemmas:

LEMMA 2.1: $HJ(l,d+1,k) \leq HJ(l,1,k) + HJ(l,d,k^{l^{HJ(l,1,k)}})$

Proof (*): Define n_1 and n_2 as the first and second term of the right-hand side of the equation, respectively. Let $n = n_1 + n_2$, $C = \{c_1,\ldots,c_k\}$ and let $c : W_n(L) \rightarrow C$ be an arbitrary k-coloring of $W_n(L)$. Next, define the functions c_v, for $v \in W_{n_2}(L)$, and $\tilde{c}$ as

$$\begin{aligned} c_v &: W_{n_1}(L) \rightarrow C : w \rightarrow c(wv), \\ \tilde{c} &: W_{n_2}(L) \rightarrow C^{W_{n_1}(L)} : v \rightarrow c_v, \end{aligned}$$

where $C^{W_{n_1}(L)}$ represents all the functions from $W_{n_1}(L)$ to C. $W_{n_1}(L)$ contains l^{n_1} strings; therefore, there are $k^{l^{n_1}}$ such functions. Meaning, that $\tilde{c}$ can be seen as a coloring of a n_2-dimensional hypercube with $k^{l^{n_1}}$ colors. Thus, there exists a monochromatic d-parameter word V (of length n_2), that is, all the strings represented by V are mapped onto the same function, say c'_v. This function c'_v is a coloring of a n_1-dimensional hypercube with k colors; therefore, there exist a monochromatic 1-parameter word W (of length n_1). As a result, WV is monochromatic $d+1$-parameter word (or subspace) under the function c.

Q.E.D.

LEMMA 2.2: $HJ(l+1, 1, k+1) \leq HJ(l, 1 + HJ(l+1, 1, k), k+1)$

Proof (*): Let n be equal to the right-hand side of the equation, let L be an alphabet with l letters and let $c : W_n(L \cup \{z\}) \to \{c_1, \ldots, c_{k+1}\}$ be an arbitrary $k+1$-coloring of an n-dimensional hypercube with width $l+1$. Define

$$c' : W_n(L) \to \{c_1, \ldots, c_{k+1}\} : w \to c(w).$$

Then, by definition of n, there exists a monochromatic $1 + HJ(l+1, 1, k)$-parameter word V (under c'), that is, all the strings represented by V (over the alphabet L) are mapped onto the same color, say c_i.
Define $C = \{c_1, \ldots, c_{k+1}\} - \{c_i\}$. We distinguish two cases: (i) c assigns color c_i to at least 1 string s represented by V (over the alphabet $L \cup \{z\}$) and this string s contains at least 1 letter z. Then, replace z by X in V to find a monochromatic 1-parameter word (under c). (ii) c never assigns color c_i to a string s that is represented by V and that contains at least 1 letter z. Then, replace 1 parameter of V by z (arbitrary) to find the $HJ(l+1, 1, k)$-parameter word V'. Now, c maps all the strings represented by V' (over $L \cup \{z\}$) onto C (where $|C| = k$). These strings form a $HJ(l+1, 1, k)$-dimensional hypercube of width $l+1$; therefore, there exists a monochromatic 1-parameter word W (under c of length $HJ(l+1, 1, k)$). If we now substitute W into the $HJ(l+1, 1, k)$ parameters of V' we obtain the required combinatorial line.

Q.E.D.

THEOREM 2.1: All Hales-Jewett numbers $HJ(l, d, k)$ are finite.

Proof: Suppose that some set S_0 of HJ-numbers are infinite. Then, let S_1 be the subset of S where l is minimal, let S_2 be the subset of S_1 where d is minimal and let S_3 be the subset of S_2 where k is minimal. Take an arbitrary number $HJ(l, d, k)$ from S_3. Clearly, l or k cannot be equal to 1, hence, $HJ(l, d, k)$ can be written as the left-hand side of lemma 2.1 or 2.2. Thus, the right-hand side has to be infinite as well. But, by construction of S_3, this is impossible.

Q.E.D.

Exercises 2.1: Prove the following two statements:

- Playing Tic-Tac-Toe in an 18-dimensional (or higher) space can never result in a draw.
- Prove the Bartel Van der Waerden Theorem which states that for any $l > 0$, there exists an N finite such that for any k-coloring c of $[1, N]$, there exists a monochromatic arithmetic progression of length l, that is, an a, b for which $a, a + b, \ldots, a + (l - 1)b$ have the same color [Hint: Choose $N = (l - 1)HJ(l, 1, k)$, $n = N/(l - 1)$ and define $c' : W_n(L) \to \{1, \ldots, k\} : w_1 w_2 \ldots w_n \to c(\sum_i w_i)$].