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Approach spaces.

The missing link in the topology-uniformity-metric triad. Oxford Mathematical Monographs. Oxford Science Publications. *The Clarendon Press, Oxford University Press, New York,* 1997. x+253 pp. ISBN: 0-19-850030-0

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FEATURED REVIEW.

This book is a landmark in the history of general topology. The author developed approach spaces in a series of papers from 1987 onwards and the book contains the results from those papers, presented in a systematic and definitive manner. Approach spaces form a supercategory of the categories of topological spaces and metric spaces, and the main purpose for their introduction was to fill some gaps concerning categorical aspects of metrizable spaces (see below for details). They not only solved these fundamental problems but also turned out to be interesting objects of study for their own sake. Moreover, they are useful in many areas such as probability theory, functional analysis, function spaces, hyperspaces, and probabilistic metric spaces. They also provide a proper setting for a unified treatment of certain topological and metric properties in the theory of approximation.

Let us discuss a motivation. Suppose (X,d) is a metric space and $\delta_d(x,A)$ is the distance between $x\in X$ and $A\subset X$ defined by $\inf\{d(x,a)\colon a\in A\}$. We say that x is near A if $\delta_d(x,A)=0$. About ninety years back, F. Riesz showed that the concept of a continuous function can be defined by using this nearness relation between points and subsets. A function f from one metric space X to another metric space Y is continuous at $c\in X$ if whenever c is near $A\subset X$ in X, $f(c)$ is near $f(A)$ in Y. Most teachers and texts of calculus motivate continuity by an intuitive statement such as "when x is near c, $f(x)$ is near $f(c)$". This intuitive statement, which is covariant, is not precise. To make it precise with the use of metrics, one arrives at the usual contravariant ϵ-δ definition. But this contravariant definition is neither easy to teach nor easy for a student to understand. This concern has been debated in recent issues of the Notices of the AMS. When we understand that continuity is a topological concept and not a metric one, then the situation becomes simple. Riesz's definition is unique in mathematics; it is both intuitive and rigorous at the same time. Of course, the metric definition is useful and students need to understand it [see P. Cameron, J. G. Hocking and S. A. Naimpally, Amer. Math. Monthly **81** (1974), 739–745; MR0349910]. From the above discussion, it is clear that the distance δ_d is more valuable than the metric d. The same conclusion is reached by Lowen from the categorical point of view.

The category $\mathbf{TOP}$, of topological spaces and continuous maps, is well behaved with respect to subspaces, products, quotients, coproducts, etc. On the other hand, the category $\mathbf{MET}$, of metric spaces and nonexpansive maps, is stable only under the formation of subspaces and finite products. Even when metric spaces are used in analysis (e.g. Banach spaces), topology provides the framework for most of the basic concepts such as continuity, convergence, compactness, etc. This is partially remedied by the use of the extended pseudometrics, i.e. the category $p\mathbf{MET}^\infty$. The new category still does not behave well with respect to initial structures. Weil introduced the concept of a uniform space, which makes it possible to extend some metric concepts such as total boundedness and completeness. But we lose numerical information that was available earlier. Consider the following diagram:

$$\begin{array}{ccccc} p\mathbf{MET}^\infty & \to & ? & \to & ? \\ \downarrow & & \downarrow & & \downarrow \\ pm\mathbf{UNIF} & \to & \mathbf{UNIF} & \to & q\mathbf{UNIF} \\ \downarrow & & \downarrow & & \downarrow \\ pm\mathbf{TOP} & \to & \mathbf{CREG} & \to & \mathbf{TOP} \end{array}$$

In the second row, we have $q\mathbf{UNIF}$, the category of quasi-uniform spaces and uniformly continuous maps, $\mathbf{UNIF}$, of uniform spaces, and $pm\mathbf{UNIF}$, of pseudometrizable uniformities. In the third row, we have $\mathbf{TOP}$, and its subcategories $\mathbf{CREG}$, of completely regular spaces, and $pm\mathbf{TOP}$, of pseudometrizable topological spaces. The first row gives numerical information, the second uniform information (entourages) and the third local information (neighborhoods). The horizontal arrows stand for embedding functors and the vertical arrows for the forgetful functors. The two question marks show that the diagram is incomplete; do there exist structures which fill the gaps such that the two new vertical arrows represent forgetful functors and the two new horizontal arrows represent the embedding functors? Lowen shows that approach spaces fill the gaps in an admirable manner. In addition to the above motivation, Lowen points out several other considerations which call for a new structure: (1) products of metric spaces, (2) analysis versus numerical analysis and approximation theory, (3) topological properties versus metric properties, (4) quantification of topological properties such as the Kuratowski or Hausdorff measure of non-compactness.

The central idea in *Approach spaces* is that of a distance δ, which is a function on $X\times 2^X$ to $[0,\infty]$. And the interesting fact is that a distance can be defined not only in a metric space, as we saw above, but also in a topological space, a uniform space, etc. Of course, the abstract distance to be defined should satisfy some very simple general conditions: (D1) For each x in X, $\delta(x,\{x\})=0$. (D2) For each x in X, $\delta(x,\emptyset)=\infty$. (D3) For each x in X and for all A,B in 2^X, $\delta(x,A\cup B)=\min\{\delta(x,A),\delta(x,B)\}$. (D4) For each x in X, for each A in 2^X and each ϵ in $[0,\infty]$, $\delta(x,A)\le\delta(x,A^{(\epsilon)})+\epsilon$, where $A^{(\epsilon)}=\{x\in X\colon \delta(x,A)\le\epsilon\}$. We have already seen how a distance arises from a metric. Lowen shows how we can define a distance in a topological space or a uniform space. For example, in a topological space, we define $\delta(x,A)=0$ if $x\in \mathrm{cl}(A)$, $\delta(x,A)=\infty$ otherwise. The distance δ so defined satisfies (D1)–(D4) and is compatible with the topology. If δ is a distance on X, then (X,δ) is called an approach space. If (X,δ) and (X,δ') are approach spaces, a function $f\colon X\to Y$ is called a contraction if for each x in X and for each $A\subset X$, $\delta(f(x),f(A))\le\delta(x,A)$. The approach spaces form the objects and contractions form the morphisms of a construct denoted by $\mathbf{AP}$. A topological space can be described in several equivalent ways: open sets, closed sets, Kuratowski closure operator, etc. Similarly, approach spaces can be defined not only in terms of a distance but also in terms of limit operators, approach systems, gauges, towers, hull operators, regular function frames, etc. Lowen describes these and shows their equivalence. A clear diagram is provided showing various connections. The first chapter ends by showing that $\mathbf{AP}$ is a topological construct.

The second chapter deals with topological approach spaces. In these spaces, distances arise from topologies as described above. The functor $\mathbf{TOP}\to\mathbf{AP}$ is a full concrete embedding. Moreover, $\mathbf{TOP}$ is simultaneously bireflectively and bicoreflectively embedded in $\mathbf{AP}$. In the third chapter, we find metric approach spaces. Here the distances arise from extended pseudo-quasimetrics. The category of extended pseudo-quasimetric spaces as objects and nonexpansive maps as morphisms $pq\mathbf{MET}^\infty$ is also nicely embedded in $\mathbf{AP}$. Uniform approach spaces are dealt with in the fourth chapter. These are subspaces of products of ∞p-metric approach spaces in $\mathbf{AP}$. They form the epireflective hull, denoted by $\mathbf{UAP}$, of $p\mathbf{MET}^\infty$ in $\mathbf{AP}$. Their convergence in $\mathbf{UAP}$ and their relationship with $\mathbf{UNIF}$ are discussed. The two gaps (question marks) in the diagram given above are now filled. The new first row looks like

$$p\mathbf{MET}^\infty\to\mathbf{UAP}\to\mathbf{AP}$$

and the diagram is complete!

Chapter five gives canonical examples: spaces of measures, function spaces, hyperspaces, probabilistic metric spaces and spaces of random variables. Subsequent chapters give approach properties (compactness, connectedness, completeness), completion (construction, and an example of a function space) and compactification (construction, βX).

For purposes of reference, Appendix A provides information on basic categorical concepts, topological constructs, reflective and coreflective subcategories. Appendix B contains (probabilistic) metric spaces, convergence spaces, quasi-uniform and uniform spaces and proximity spaces. There is a bibliography of 117 items and an index.

The book is very carefully written, and, with hindsight, the author has presented the material in a simpler manner than that used in his original papers. Although the subject matter goes deep into abstract territory, several examples are provided which make the book easy to read. The book deserves a place on the shelves of mathematicians, especially topologists, who are interested in fundamentals.

Reviewed by S. A. Naimpally