The stability of θ-methods for systems of delay differential equations*

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This paper deals with the numerical solution of initial value problems for systems of differential equations with a delay argument. We investigate the stability of adaptations of the θ-methods in the numerical solution of these problems. We assess the stability of the adaptations under consideration by analyzing their stability behaviour in the solution of the test equation $U'(t) = LU(t) + MU(t-\tau) \quad (t \geq 0)$, where L, M denote constant complex matrices, and $\tau > 0$.

Keywords: delay differential equations, numerical solution, θ-methods, stability.

Subject classification: AMS 65L20.

1. Introduction

1.1. ADAPTATION OF THE θ-METHODS TO DELAY DIFFERENTIAL EQUATIONS

This paper deals with the numerical solution of initial value problems for systems of delay differential equations,

$$U'(t) = f(t, U(t), U(t-\tau)) \quad (t \geq 0), \quad U(t) = g(t) \quad (-\tau \leq t \leq 0), \tag{1.1}$$

where f, g denote given vector-valued functions, τ is a given real number with $\tau > 0$, and $U(t)$ is unknown (for $t > 0$). We are interested in the case where (1.1) is stiff.

For the numerical solution of (1.1) we consider adaptations of the θ-methods. The θ-methods are well-known numerical methods for solving problems (1.1) without a delay argument $U(t-\tau)$. If $h > 0$ denotes a given stepsize, the gridpoints t_n are given by $t_n = nh$ $(n = 0, 1, 2, \ldots)$ and $u_0 = g(t_0)$, then adaptation

* This work was supported by the Netherlands Organization for Scientific Research (NWO)
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of the so-called *1-stage* θ-methods to problems of type (1.1) gives rise to the following formula defining approximations u_n to $U(t_n)$,

$$u_n = u_{n-1} + hf(t_{n-1} + \theta h, \theta u_n + (1-\theta)u_{n-1}, v_n) \quad (n \geq 1). \tag{1.2.a}$$

Adaptation of the so-called *2-stage* θ-methods gives rise to the formula

$$u_n = u_{n-1} + h\{\theta f(t_n, u_n, v_n) + (1-\theta) f(t_{n-1}, u_{n-1}, v_{n-1})\} \quad (n \geq 1). \tag{1.3.a}$$

Here, $\theta \in [0, 1]$ denotes a real parameter that specifies the method. Further, v_n in (1.2.a) denotes an approximation to $U(t_{n-1} + \theta h - \tau)$, whereas v_n in (1.3.a) denotes an approximation to $U(t_n - \tau)$.

In order to give the definitions for v_n in (1.2.a), (1.3.a) that we deal with in this paper, we write $\tau = (m - \delta)h$ with integer $m \geq 1$ and $\delta \in [0, 1)$. Further, we put $\widehat{t}_j = t_{j-1} + \theta h$, $\widehat{u}_j = \theta u_j + (1-\theta)u_{j-1}$ (for $j = 1, 2, 3, \ldots$). Then, v_n in (1.2.a) is defined by

$$v_n = \delta \widehat{u}_{n-m+1} + (1-\delta)\widehat{u}_{n-m} \quad (\text{whenever } n \geq m+1), \tag{1.2.b}$$

and v_n in (1.3.a) is defined by

$$v_n = \delta u_{n-m+1} + (1-\delta)u_{n-m} \quad (\text{whenever } n \geq m). \tag{1.3.b}$$

Thus, in the case of the 1-stage θ-methods the approximations v_n are obtained from linear interpolation at points $\widehat{t}_j$ using values $\widehat{u}_j$ (with $j \leq n$), whereas in the case of the 2-stage θ-methods the v_n are obtained from linear interpolation at (grid)points t_j using values u_j (with $j \leq n$).

We note that, for the purposes of our paper, we do not define v_n in (1.2.a), (1.3.a) for $n \leq m$ and $n \leq m - 1$, respectively.

The adaptation of the 1-stage θ-methods described above is equivalent to the adaptation that has been proposed for these methods in [6]. The adaptation of the 2-stage θ-methods has already been considered often in the literature (cf. e.g. [1], [3], [6], [13], [14], [19]). Further, the interpolation formulas (1.2.b), (1.3.b) are both of the new type of interpolation procedures recently considered in [9] for adapting general Runge-Kutta methods to problems of type (1.1), and consequently methods (1.2), (1.3) belong to the general class of numerical methods for (1.1) that is investigated in [9].

For a 1-stage θ-method, a natural, alternative adaptation to (1.1) is obtained when v_n in (1.2.a) is computed from linear interpolation at points t_j using values u_j (with $j \leq n$), just as in the case of the 2-stage θ-methods. However, for reasons of stability (cf. section 1.2), we consider in this paper the adaptation formulated above instead of dealing with this alternative adaptation.

1.2. STABILITY ANALYSIS OF METHODS (1.2),(1.3)

The aim of our paper is to gain insight into the stability of methods (1.2), (1.3) in the numerical solution of general systems (1.1). For that purpose, we investigate the stability behaviour of the methods in the numerical solution of the test equation

$$U'(t) = LU(t) + MU(t-\tau) \quad (t \geq 0), \tag{1.4}$$

where L, M denote constant complex $d \times d$–matrices and $\tau > 0$.

Define $X = hL$, $Y = hM$ and let I denote the $d \times d$ identity matrix. Application of methods (1.2), (1.3) in case of the linear system (1.4) yields, for both methods, the following linear recurrence relation for the approximations u_n $(n \geq m+1)$,

$$\begin{aligned}(I - \theta X)u_n = \; & (I + (1-\theta)X)u_{n-1} + \delta\theta Y u_{n-m+1} + \\ & +(\delta(1-\theta) + (1-\delta)\theta)Y u_{n-m} + (1-\delta)(1-\theta)Y u_{n-m-1} \,.\end{aligned} \tag{1.5}$$

We have

DEFINITION 1.1

Let (X, Y) be a given pair of complex $d \times d$–matrices. Then, process (1.5) is called *stable at* (X, Y) if

(i) the matrix $(I - \theta X - \delta\theta Y)$ is invertible whenever $0 \leq \delta < 1$,

(ii) any solution $u_0, u_1, u_2, \ldots$ to (1.5) satisfies $\lim_{n\to\infty} u_n = 0$ whenever $m \geq 1$, $0 \leq \delta < 1$.

In the literature, many authors have dealt with the scalar case ($d = 1$) of test equation (1.4) in order to arrive at conclusions about the stability of numerical methods for delay differential equations (cf. e.g. [1], [3], [6]–[10], [13], [19]–[21]). From these investigations, a complete characterization for the set S_θ of all pairs of complex *numbers* (x, y) at which process (1.5) is stable can easily be obtained, cf. [13]. Further, the question has been studied whether or not, for a given θ, the condition $H \subset S_\theta$ is fulfilled, where $H = \{(x, y) \mid x \in \mathbb{C},\ y \in \mathbb{C},\ \mathrm{Re}(x) < -|y|\}$. The reason for considering this condition lies in the fact that for equation (1.4) with $d = 1$ it is known (cf. e.g. [1], [21]) that

$$\mathrm{Re}(\lambda) < -|\mu| \;\Rightarrow\; \lim_{t\to\infty} U(t) = 0 \text{ (whenever } \tau > 0) \;\Rightarrow\; \mathrm{Re}(\lambda) \leq -|\mu| \,,$$

where we have written $L = (\lambda)$, $M = (\mu)$. The condition $H \subset S_\theta$ can be viewed as a generalization of the concept of A-stability to the case of delay differential equations. It follows from [3] that for process (1.5) one has $H \subset S_\theta$ if and only if

$\theta \in [\frac{1}{2}, 1]$ (see also [9], [10], [13], [19]). This is clearly in complete correspondence with the A-stability of the underlying θ-methods.

We remark that for the adaptation of a 1-stage θ-method by linear interpolation at the gridpoints (cf. end of section 1.1) it has been shown (cf. [13], [19], [20], also [8], [10]) that the stability condition $H \subset S_\theta$ is always violated (whenever $\theta < 1$).

The general case of test equation (1.4) seems not to have been studied in the literature so far. Some results on the asymptotic stability of process (1.5) and equation (1.4) can be obtained from the literature for cases where the matrices L, M are of a special type. In [4] results have been derived relevant to the case where L, M are both real and symmetric. In [14] results have been obtained when L, M are diagonal and reverse diagonal, respectively. Further, if L, M are simultaneously diagonalizable, then the results from the scalar case of (1.4) can immediately be generalized.

In this paper we shall consider the general case of (1.4), i.e., arbitrary dimension d and arbitrary matrices L, M.

1.3. SCOPE OF OUR PAPER

In section 2 we derive a complete characterization for the set of all pairs of complex $d \times d$–matrices (X, Y) at which process (1.5) is stable. This generalizes the known characterization for $d = 1$ (cf. [13]) to the general case $d \geq 1$.

In section 3 we obtain a new and simple criterion on the matrices L, M such that all exact solutions U to test equation (1.4) satisfy $U(t) \to 0$ for $t \to \infty$ (whenever $\tau > 0$). This generalizes the criterion of [21], which dealt with the case where $d = 1$.

The results from [4], [14] on the stability of (1.5), (1.4) follow easily from our results in sections 2, 3.

In section 4 we assess the stability of the numerical methods (1.2), (1.3) by comparing the stability results from sections 2, 3. Further, we consider adaptation of the θ-methods by using the general interpolation procedure from [9]. Finally, we give some references for stiff systems of delay differential equations that arise in mathematical modelling.

2. The stability of process (1.5)

Denote for any matrix A its determinant by $\det[A]$, its spectrum by $\sigma[A]$, and its spectral radius by $\rho[A]$.

Let $\theta \in [0, 1]$ be given, and let (X, Y) be a given pair of complex $d \times d$–matrices. Define $P(z; \delta) = (\delta z + 1 - \delta)(\theta z + 1 - \theta)Y$, $Q(z) = z(I - \theta X) - (I + (1 - \theta)X)$

(whenever $z \in \mathbb{C}$, $0 \le \delta < 1$). It easily follows from well-known results on linear recurrence relations (cf. e.g. [12]) that process (1.5) is stable at (X, Y) if and only if

$$(I - \theta X - \delta\theta Y) \text{ is invertible (whenever } 0 \le \delta < 1), \quad (2.1.a)$$

$$\det[z^m Q(z) - P(z;\delta)] = 0 \Rightarrow |z| < 1 \quad (2.1.b)$$
$$(\text{whenever integer } m \ge 1,\ 0 \le \delta < 1).$$

Consider the statements

$(I - \theta X - \delta\theta Y)$ is invertible (whenever $0 \le \delta < 1$), $Q(z)$ is invertible (whenever $|z| \ge 1$), $\sup_{|z|=1} \rho[Q(z)^{-1} P(z;\delta)] < 1$ (whenever $0 \le \delta < 1$), (2.2.a)

$(I - \theta X - \delta\theta Y)$ is invertible (whenever $0 \le \delta < 1$), $Q(z)$ is invertible (whenever $|z| \ge 1$), $\sup_{|z|=1} \rho[Q(z)^{-1} P(z;\delta)] \le 1$ (whenever $0 \le \delta < 1$). (2.2.b)

By application of a general theorem that was derived in [7] on conditions of type (2.1.b), we immediately obtain

LEMMA 2.1

The following implications hold,

$$(2.2.a) \Rightarrow \text{process (1.5) is stable at } (X, Y) \Rightarrow (2.2.b).$$

In the following we investigate the statements (2.2.a), (2.2.b). Consider the well-known stability region S^* given by

$$S^* = \{\zeta \mid \zeta \in \mathbb{C},\ \left|\frac{1 + (1-\theta)\zeta}{1 - \theta\zeta}\right| < 1\}.$$

LEMMA 2.2

Conditions (i), (ii) are equivalent, where
(i) $\sigma[X] \subset S^*$,
(ii) $(I - \theta X)$ is invertible, $Q(z)$ is invertible (whenever $|z| \ge 1$).

Proof

The matrix $Q(z)$ is invertible (whenever $|z| \ge 1$) if and only if

$$\left(\frac{z-1}{\theta z + 1 - \theta} I - X\right) \text{ is invertible (whenever } |z| \ge 1,\ \theta z + 1 - \theta \ne 0).$$

One easily verifies that

$$\{\zeta = \frac{z-1}{\theta z + 1 - \theta}, \ \theta z + 1 - \theta \neq 0\} \Longleftrightarrow \{z = \frac{1+(1-\theta)\zeta}{1-\theta\zeta}, \ 1 - \theta\zeta \neq 0\}, \quad (2.3)$$

and it follows that $Q(z)$ is invertible (whenever $|z| \geq 1$) if and only if $(\zeta I - X)$ is invertible (whenever $\zeta \notin S^*$, $1 - \theta\zeta \neq 0$). This yields the equivalence in the lemma. □

Let

$$\Gamma = \{\zeta \mid \zeta \in \mathbb{C}, \ \left|\frac{1+(1-\theta)\zeta}{1-\theta\zeta}\right| = 1\}.$$

LEMMA 2.3

Assume $\sigma[X] \subset S^*$. Then for all $\delta \in [0, 1)$,

$$\sup_{|z|=1} \rho[Q(z)^{-1}P(z;\delta)] \ \leq \ \sup_{\zeta\in\Gamma} \rho[(\zeta I - X)^{-1}Y].$$

Moreover, if $\delta = 0$, then the above inequality is an equality.

Proof

For $z \in \mathbb{C}$ with $|z| = 1$, $\theta z + 1 - \theta \neq 0$ we have

$$\begin{aligned} \rho[Q(z)^{-1}P(z;\delta)] &= |\delta z + 1 - \delta| \cdot \rho[(\tfrac{z-1}{\theta z+1-\theta} I - X)^{-1}Y] \\ &\leq \rho[(\tfrac{z-1}{\theta z+1-\theta} I - X)^{-1}Y], \end{aligned}$$

where the last inequality is an equality if $\delta = 0$. Further, $\rho[Q(z)^{-1}P(z;\delta)] = 0$ whenever $\theta z + 1 - \theta = 0$. From this and the equivalence (2.3), the statements in the lemma follow. □

LEMMA 2.4

Assume $\sigma[X] \subset S^*$. Then

$$\rho[(\zeta I - X)^{-1}Y] \ \leq \ \sup_{\zeta\in\Gamma} \rho[(\zeta I - X)^{-1}Y] \quad \text{whenever } \zeta \notin S^* \,.$$

For brevity, we omit the proof of lemma 2.4. We remark that it follows from the maximum principle (cf. e.g. [5]) and the fact that the eigenvalues of $(\zeta I - X)^{-1}Y$ are algebraic functions (cf. e.g. [11]) of the complex variable ζ.

LEMMA 2.5

Assume $\sigma[X] \subset S^*$ and $\sup_{\zeta \in \Gamma} \rho[(\zeta I - X)^{-1}Y] \leq 1$. Then the matrix $(I - \theta X - \delta\theta Y)$ is invertible (whenever $0 \leq \delta < 1$).

Proof

Assume $\theta \neq 0$. Since $\frac{1}{\theta} \notin S^*$, we have that $(I - \theta X - \delta\theta Y)$ is invertible if and only if $(I - \delta(\frac{1}{\theta}I - X)^{-1}Y)$ is invertible. Lemma 2.4 implies $\rho[(\frac{1}{\theta}I - X)^{-1}Y] \leq 1$, and the statement follows. □

By a combination of lemmas 2.1, 2.2, 2.3 and 2.5 we arrive at the main result of this section.

THEOREM 2.6

Consider the statements

$$\sigma[X] \subset S^* \quad , \quad \sup_{\zeta \in \Gamma} \rho[(\zeta I - X)^{-1}Y] < 1, \quad (2.4.a)$$
$$\sigma[X] \subset S^* \quad , \quad \sup_{\zeta \in \Gamma} \rho[(\zeta I - X)^{-1}Y] \leq 1. \quad (2.4.b)$$

Then, (2.4.a) $\Rightarrow$ process (1.5) is stable at (X, Y) $\Rightarrow$ (2.4.b).

3. The stability of test equation (1.4)

Define $H^* = \{\zeta \mid \zeta \in \mathbb{C},\ \mathrm{Re}(\zeta) < 0\}$. Let L, M be given complex $d \times d$–matrices and consider the statements

$$\text{all exact solutions } U \text{ to (1.4) satisfy } \lim_{t \to \infty} U(t) = 0 \text{ (whenever } \tau > 0), \quad (3.1)$$

$$\sigma[L] \subset H^*,\ \rho[(\zeta I - L)^{-1}M] < 1 \text{ (whenever } \mathrm{Re}(\zeta) = 0,\ \zeta \neq 0), \text{ and } -1 \notin \sigma[L^{-1}M]. \quad (3.2)$$

Then we have

THEOREM 3.1

Statements (3.1), (3.2) are equivalent.

Proof

From [2] we immediately obtain that (3.1) holds if and only if

$$\det[\zeta I - L - e^{-\tau\zeta}M] = 0 \quad \Rightarrow \quad \mathrm{Re}(\zeta) < 0 \quad (\text{whenever } \tau > 0). \tag{3.3}$$

In the following we show that (3.2) $\Longleftrightarrow$ (3.3).

1. Assume (3.2). Then (cf. lemma 2.4 with $\theta = \frac{1}{2}$)

$$\rho[(\zeta I - L)^{-1}M] \leq 1 \quad \text{whenever } \mathrm{Re}(\zeta) > 0.$$

By using that

$$\det[\zeta I - L - e^{-\tau\zeta}M] = 0 \Longleftrightarrow e^{\tau\zeta} \in \sigma[(\zeta I - L)^{-1}M] \quad (\text{whenever } \mathrm{Re}(\zeta) \geq 0),$$

it easily follows that (3.3) holds.

2. Assume (3.3). We first prove that all eigenvalues λ of L satisfy $\mathrm{Re}(\lambda) \leq 0$. Suppose that there exists $\lambda \in \sigma[L]$ with $\mathrm{Re}(\lambda) > 0$. Let Λ be a positively oriented circle centered at λ such that $\mathrm{Re}(\zeta) > 0$ and $(\zeta I - L)$ is invertible (whenever $\zeta \in \Lambda$). For any complex valued function f that is defined, continuous and nonzero on Λ, we denote by $[\arg f(\zeta)]_\Lambda$ the increase of the argument of f along Λ (cf. [5]). Note that, since Λ is closed, $[\arg f(\zeta)]_\Lambda$ is equal to an integer multiple of 2π. Let $\tau > 0$ be given such that $\rho[e^{-\tau\zeta}(\zeta I - L)^{-1}M] < 1$ whenever $\zeta \in \Lambda$. For $\alpha \in [0,1]$ define $h_\alpha(\zeta) = \det[I - \alpha e^{-\tau\zeta}(\zeta I - L)^{-1}M]$. Since $h_\alpha(\zeta) \neq 0$ for any $\zeta \in \Lambda$, we have that $[\arg h_\alpha(\zeta)]_\Lambda$ is defined and equal to an integer multiple of 2π. Moreover,

$$[\arg h_\alpha(\zeta)]_\Lambda = \frac{1}{i}\int_\Lambda \{h'_\alpha(\zeta)/h_\alpha(\zeta)\}\, d\zeta$$

depends continuously on α. Consequently, $[\arg h_1(\zeta)]_\Lambda = [\arg h_0(\zeta)]_\Lambda = 0$. Hence,

$$[\arg \det[I - e^{-\tau\zeta}(\zeta I - L)^{-1}M]\ ]_\Lambda = 0,$$

and

$$[\arg \det[\zeta I - L - e^{-\tau\zeta}M]\ ]_\Lambda = [\arg \det[\zeta I - L]\ ]_\Lambda\ .$$

Application of the argument principle yields that $\det[\zeta I - L - e^{-\tau\zeta}M] = 0$ for some ζ in the interior of Λ, but this contradicts (3.3). Therefore, all eigenvalues λ of L satisfy $\mathrm{Re}(\lambda) \leq 0$.

We complete the proof of theorem 3.1 by showing that (3.2) holds. Define $\sigma_0[L] = \{\lambda \mid \lambda \in \sigma[L],\ \mathrm{Re}(\lambda) = 0\}$. It easily follows from (3.3) that

(i) $\mu \in \sigma[(\zeta I - L)^{-1}M] \Rightarrow |\mu| \neq 1 \quad (\text{whenever } \mathrm{Re}(\zeta) = 0,\ \zeta \neq 0,\ \zeta \notin \sigma_0[L])$.

Further,

(ii) $$\rho[(\zeta I - L)^{-1}M] \to 0 \quad (\text{for } \mathrm{Re}(\zeta) = 0,\ |\zeta| \to \infty).$$

Let $\lambda_0 \in \sigma_0[L]$ be such that $|\lambda_0| \geq |\lambda|$ (whenever $\lambda \in \sigma_0[L]$). Suppose $\mathrm{Im}(\lambda_0) \leq 0$, and consider the polynomial p given by $p(\mu;\zeta) = \det[\mu(\zeta I - L) - M]$. Since $\rho[(\zeta I - L)^{-1}M]$ is a continuous function of ζ on $\{\zeta \mid \mathrm{Re}(\zeta) = 0,\ \zeta \notin \sigma_0[L]\}$, the conditions (i), (ii) imply that all zeros μ of $p(\mu;\zeta)$ satisfy $|\mu| < 1$ whenever $\mathrm{Re}(\zeta) = 0$, $\mathrm{Im}(\zeta) < \mathrm{Im}(\lambda_0)$. By considering $\mathrm{Re}(\zeta) = 0$, $\mathrm{Im}(\zeta) \uparrow \mathrm{Im}(\lambda_0)$, it can be seen that $p(\mu;\lambda_0) \equiv 0$. However, this contradicts (3.3). Analogously, one obtains a contradiction when $\mathrm{Im}(\lambda_0) > 0$. Therefore, $\sigma_0[L]$ is empty. It follows that all eigenvalues λ of L satisfy $\mathrm{Re}(\lambda) < 0$, and $\rho[(\zeta I - L)^{-1}M] < 1$ (whenever $\mathrm{Re}(\zeta) = 0$, $\zeta \neq 0$). Finally, (3.3) yields that $\det[L + M] \neq 0$, and consequently, $-1 \notin \sigma[L^{-1}M]$. □

Remark 3.2

The equivalence (3.2) $\Longleftrightarrow$ (3.3) can be viewed as a continuous analogue to theorem 1.1 in [7].

A useful corollary to theorem 3.1 is

COROLLARY 3.3

Consider the statements

$$\sigma[L] \subset H^* \quad , \quad \sup_{\mathrm{Re}(\zeta)=0} \rho[(\zeta I - L)^{-1}M] < 1, \quad (3.4.a)$$
$$\sigma[L] \subset H^* \quad , \quad \sup_{\mathrm{Re}(\zeta)=0} \rho[(\zeta I - L)^{-1}M] \leq 1. \quad (3.4.b)$$

Then, (3.4.a) $\Rightarrow$ (3.1) $\Rightarrow$ (3.4.b).

4. Concluding remarks

4.1. COMPARING THE STABILITY OF PROCESS (1.5) TO THE STABILITY OF TEST EQUATION (1.4)

In view of corollary 3.3 it is natural to consider the following definition.

DEFINITION 4.1

Process (1.5) is called *stable* if it is stable at (X, Y) whenever $\sigma[X] \subset H^*$ and $\sup_{\mathrm{Re}(\zeta)=0} \rho[(\zeta I - X)^{-1}Y] < 1$.

From theorem 2.6 and lemma 2.4 we immediately obtain

THEOREM 4.2

Process (1.5) is stable whenever $\theta \in [\frac{1}{2}, 1]$.

Theorem 4.2 shows that, in essence, it holds that *if* the linear system (1.4) is asymptotically stable (whenever $\tau > 0$), and $\theta \in [\frac{1}{2}, 1]$, *then* methods (1.2), (1.3), in case of (1.4), always yield sequences of approximations that are asymptotically stable whenever $h > 0$.

4.2. USING A GENERAL INTERPOLATION PROCEDURE

In [9] a general interpolation procedure was proposed for adapting the class of Runge-Kutta methods to delay differential equations. Considering this procedure in the case of the 1-stage and 2-stage θ-methods, we obtain the following formulas for the approximations v_n in (1.2.a), (1.3.a),

$$v_n = \sum_{k=-r}^{s} L_k(\delta)\widehat{u}_{n-m+k} \ , \tag{1.2.b$'$}$$

$$v_n = \sum_{k=-r}^{s} L_k(\delta)u_{n-m+k} \ , \tag{1.3.b$'$}$$

respectively. Here r, s denote given integers with $0 \leq r \leq s \leq r+2$, and

$$L_k(\delta) = \prod_{\substack{j=-r \\ j \neq k}}^{s} \left(\frac{\delta - j}{k - j}\right) \qquad (k = -r, \ldots, s).$$

The interpolation formulas (1.2.b), (1.3.b) from section 1.1 are given by the special case $(r, s) = (0, 1)$ of (1.2.b$'$), (1.3.b$'$), respectively.

For the adaptations of 1-stage and 2-stage θ-methods by the general formulas (1.2.b$'$), (1.3.b$'$) a similar stability analysis can be carried out as performed in this paper in the case of (1.2.b), (1.3.b). Using lemma 2.5 from [9], one easily verifies that theorems 2.6, 4.2 remain valid in case of these general interpolation formulas.

4.3. STIFF DELAY DIFFERENTIAL EQUATIONS IN MATHEMATICAL MODELLING

In this section we give some references for initial value problems for systems of delay differential equations that arise in mathematical modelling and appear to be stiff.

We call problem (1.1) *stiff* when for the tolerance under consideration explicit methods are not efficient for the numerical solution of the problem. A more precise definition is that $hL \gg 1$ or $hM \gg 1$, where L, M denote Lipschitz constants of the function f with respect to its second and third variables, respectively, and h stands for a stepsize in the numerical solution of (1.1) which only depends on the tolerance and the variation of the exact solution U.

Stiff initial value problems for systems of delay differential equations seem to arise often in immunology. A first example is given by [18]. The model derived in [18] consists of an initial value problem for a system of seven nonlinear differential equations with one constant delay. Numerical experiments show that this problem is probably stiff.

A second example is given by [16], [17]. The model in [16], [17] consists of an initial value problem for a system of ten nonlinear differential equations with five constant delays, and also appears to be stiff (cf. [17, p. 48]).

Finally, the monograph [15] contains various models from immunology which consist of initial value problems for systems of delay differential equations that are apparently stiff.

Acknowledgement

I would like to thank J. C. Butcher and G. A. Bocharov for their help with finding the examples mentioned in section 4.3.

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