5 Characterizing Planar Graphs

Definition 5.1: Two graphs \( G \) and \( H \) are homeomorphic if one can be obtained from the other by insertion or deletion of vertices of degree two or a sequence of such operations.

![Diagrams of graphs](image)

**Fig. 3:** Defining interior and exterior bridges

Definition 5.2: Interior and exterior bridges: By example, consider Figure 3 that depicts a planar embedding of the graph \( G \). A simple circuit \( C = xywz \) is denoted by the prominent lines. The pieces (a), (b), (c) and (d) are relative to \( C \): If a piece has two or more points of contact with \( C \) it is a bridge. In this case, (d) is an interior bridge and (a), (b) and (c) are exterior bridges.
Before we prove the celebrated Kuratowski theorem, one should notice the following. Since any planar graph can be embedded on a sphere, any area can be nominated the infinite area. Meaning that for any edge $xy$ of a planar graph $G$, we can draw $G$ in such a way that $xy$ bounds the infinite area.

**Theorem 5.1 (Kuratowski (1930)):** A graph is non planar if and only if it contains a subgraph homeomorphic to $K_5$ or $K_{3,3}$.

**Proof:** ($\iff$) $K_5$ and $K_{3,3}$ are non planar, so if $G$ contains a subgraph homeomorphic to $K_5$ or $K_{3,3}$ it too must be non planar.

($\Rightarrow$) Assume that the theorem is not true. Consider the set of counter examples with the smallest number of edges $E$. Let $G$ be a member of this set with the least number of vertices. Denote $E(G) = N$, clearly $G$ is non planar and does not contain a subgraph homeomorphic to $K_5$ or $K_{3,3}$. For such a graph $G$ the following statements hold:

1. $G$ is connected. Assume that $G$ is disconnected, in which case it must consist of components $c_1, c_2, \ldots, c_k$ ($k \geq 2$). Subsequently, $E(c_i)$ for each $i = 1, \ldots, k$ must be less than $N$ (the case where $E(c_i) = 0$ for all $c_i$ but one is covered by the fact that the order of $G$ is minimal). Since $G$ does not contain a subgraph homeomorphic to $K_5$ or $K_{3,3}$, neither do $c_i$ ($i = 1, 2, \ldots, k$) and hence $c_1, c_2, \ldots, c_k$ are planar. If the components are planar then $G$ too must be planar. Hence, by contradiction, $G$ must be connected.

![Fig. 4: $G$ holds a cut vertex $v$](image-url)
(2) $G$ does not contain a cut vertex, meaning we cannot disconnect $G$ by removing a single vertex $v$. Assume that $G$ contains a cut vertex $v$ and let $G^*$ be $G$ with a separation at $v$ (see Figure 4). Since $G^*$ is disconnected, it must consist of components $c_i$ ($i = 1, 2, \ldots, n$). Each $c_i$ does not contain a subgraph homeomorphic to $K_5$ or $K_{3,3}$, since $G$ does not contain such a subgraph. Therefore, since $E(c_i) < N$ ($i = 1, 2, \ldots, n$), each $c_i$ is planar. Now, $v$ can be moved, in each case, to be incident to the exterior area. In this way the components can be rejoined to give a planar $G$ and so, by contradiction, $G$ does not contain a cut vertex.

![Diagram](image)

**Fig. 5:** $G'$ holds a circuit through $x$ and $y$

(3) Let $e(x, y)$ be the edge from $x$ to $y$. Let $G'$ be $G$ with $e(x, y)$ omitted. $G'$ contains a simple circuit through $x$ and $y$. Indeed, since $G$ contains no cut vertex (due to (2)), $G'$ is connected (otherwise $x$ would be a cut vertex). Now, assume that such a circuit does not exist, in which case a vertex, $T$ say, exists of the form shown in Figure 5(a), see Menger’s Theorem. Under separation $X$ and $Y$ are formed as shown in Figure 5(b) by adding an edge $e(x, T)$ and an edge $e(y, T)$. Since $G$ contains a subgraph homeomorphic to $X$ (because $T$ is connected to $y$ and $e(x, y)$ was part of $G$) and another homeomorphic to $Y$, $X$ and $Y$ do not contain subgraphs homeomorphic to $K_5$ or $K_{3,3}$. Moreover, $E(X) < N$ and $E(Y) < N$, so $X$ and $Y$ are planar. $X$ and $Y$ can be transformed such that $e(y, T)$ and $e(x, T)$ lie incident to the
exterior area and be brought together as shown in Figure 5(c). Now, deleting
\(e(x, T)\) and \(e(y, T)\) and adding \(e(x, y)\) must therefore give a plane representation of \(G\). So, by contradiction, \(G\) must contain simple circuit from \(x\) to \(y\).

We have now deduced: (i) \(G' = G - e(x, y)\) is connected and contains a simple circuit \(C\) through \(x\) and \(y\), (ii) \(G'\) contains no subgraph homeomorphic to \(K_5\) or \(K_{3,3}\), and (iii) Since \(E(G) = N\) and \(E(G') = N - 1\), \(G'\) is planar.

Let \(G'_p\) be a planar embedding of \(G'\) and let \(C\) be a circuit through \(x\) and \(y\) such that it encloses as many areas as possible. Denote the path from \(x\) to \(y\) inclusive by \([x, y]\) and the path from \(x\) to \(y\) non inclusive by \((x, y)\). No exterior bridge can have more than one point of contact with \(C\) in \([x, y]\) or \([y, x]\), or else a circuit \(C\) could be found that encloses at least one more area.

Now, consider the interior and exterior bridges of \(G'_p\) with respect to \(C\) so that \(G\) will be non planar. There must be at least one exterior (\(E\)) and one interior (\(I\)) bridge or else \(G\) would be planar (by drawing \(e(x, y)\) as an interior resp. exterior bridge). \(E\) must have contact points \(i\) and \(j\) with \(C\), such that \(i \in (x, y)\) and \(j \in (y, x)\), and \(I\) must have at least 2 contact points \(a\) and \(b\) with \(C\) such that \(a \in (x, y)\) and \(b \in (y, x)\) (such a bridge \(I\) exists as \(e(x, y)\) could otherwise be drawn as an interior bridge). The 4 possibilities which meet this criteria are given by Figure 6 (case (C) and (D) are equivalent).

The graph drawn in scenario A has a subgraph homeomorphic to \(K_{3,3}\) (i.e., \(\{i, b, x\}, \{j, a, y\}\)). In the other 4 scenarios the interior bridge \(I\) can still be drawn as an exterior bridge. We have chosen this interior bridge \(I\) such that it cannot be drawn as an exterior bridge without violating the planarity of \(G'_p\) (indeed, if all interior bridges could be drawn on the outside then drawing \(e(x, y)\) as an interior bridge would make \(G\) planar), thus there is either another exterior bridge \(E'\) that prevents this, which implies that we end up in scenario A, with \(E'\) playing the role of \(E\) (because all exterior bridges have 1 contact point on \((x, y)\) and 1 on \((y, x)\)). Or \(I\) has at least one more contact point \(c\) that prevents us from drawing \(I\) as an outside bridge. Let us discuss these 4 scenarios one at a time.

In scenario C we need to add \(c\) on \((j, i)\). However adding \(c\) on \((x, i)\) would result in scenario A (where \(c\) plays the role of \(a\)); hence, we add \(c\) on \((j, x)\) (see Figure 7C). We know prove that this graph contains a subgraph homeomorphic to \(K_{3,3}\). The bridge \(I\) contains a vertex \(v\) such that there are 3
vertex-distinct paths from $v$ to $a, b$ and $c$. As a result, \{a, b, c\} and \{v, j, y\} form the vertices of a subgraph of $G$ which is homeomorphic to $K_{3,3}$. Similarly, in scenario $D$ we need to add $c$ on \([y, j]\) and \{a, b, c\} and \{v, j, x\} form the vertices of a subgraph of $G$ which is homeomorphic to $K_{3,3}$ (see Figure 7D). Also, in scenario $E$, $c$ needs to coincide with $x$, otherwise if $c$ was part of \((j, x)\) or \((x, i)\) and we end up in scenario $A$. As a result, \{a, b, x\} and \{v, y, i\} form the vertices of a subgraph of $G$ which is homeomorphic to $K_{3,3}$ (see Figure 7E).

In scenario $B$, we need to add two contact points $c$ and $d$ (otherwise we could still draw $I$ on the outside), one on \((j, i)\) and one on \((i, j)\). If either $c$ or $d$ do not coincide with $x$ or $y$, we end up in scenario $C$ or $D$. Thus, $c$ has to coincide with $x$ and $d$ with $y$. We distinguish 2 cases: (i) there exists a vertex $v$ in $I$ such that there are 4 vertex-distinct paths from $v$ to $a, b, c$ and $d$ (see Figure 7B2). In this case $v, a, b, c$ and $d$ form the vertices of a subgraph of $G$ which is homeomorphic to $K_5$. (ii) if there is no such vertex $v$, then 2 vertices $w_1$ and $w_2$ exist\(^5\) such that there are 5 vertex-distinct paths: one from $w_1$ to

\(^4\) Indeed, let $P$ be a path from $a$ to $b$ on $I$ and $w$ a vertex on this path, then setting $v$ equal to the first common vertex of $P$ and a path $P'$ from $c$ to $w$ in $I - \{a, b\}$ suffices.

\(^5\) Indeed, there exists a $v$ such that there are vertex-distinct paths from $v$ to $a, b$ and $c$. 

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**Fig. 6:** 5 possible scenarios
$w_2$, two from $w_1$ to $a$ and $c$, and two from $w_2$ to $b$ and $d$ (see Figure 7B1). As a result, \{w_1, b, d\} and \{a, c, w_2\} form the vertices of a subgraph of $G$ which is homeomorphic to $K_{3,3}$.

Q.E.D.

**Exercises 5.1:** On the Kuratowski Theorem:

1. Check whether the Petersen graph is planar.

2. Determine the number of non planar graphs $G$ with 6 vertices.

## 6 Maximal Planar Graphs

**Definition 6.1:** A maximal planar graph $G$ is a planar graph to which no new edge can be added without violating the planarity of $G$. A triangulation is a planar graph $G$ in which every area (region) is bounded by three edges.

**Theorem 6.1:** The following statements are equivalent for a graph $G$ with $n_G$ vertices and $e_G$ edges:

- (in $T$) Now, there is also a path from $d$ to $c$ and $G'$ is planar. Thus, the path from $d$ to $c$ has to connect to the path from $a$ to $b$ at some point. Without loss of generality we assume that this happens below $v$.  

1. $G$ is maximal planar.
2. $G$ is a triangulation.
3. $e_G = 3n_G - 6$ and $G$ is planar.

**Proof:** (1 $\Rightarrow$ 2) Suppose there is a planar representation of $G$ that contains an area $A$ bounded by 4 or more edges. Two vertices $x$ and $y$ incident to $A$ exist such that $xy \notin E(G)$, otherwise we could add a vertex in the interior of $A$ to obtain a planar representation of $K_5$. Drawing the edge $xy$ in the interior of $A$ does not violate the planarity of $G$.

(2 $\Rightarrow$ 3) $G$ is a triangulation, therefore, $3f = 2e$. Applying Euler’s equality for planar graphs we find $3n - e = 6$.

(3 $\Rightarrow$ 1) Earlier in the course, we saw that $e \leq 3n - 6$ for any planar graph $G$. Adding an edge thus violates the planarity of $G$. Q.E.D.

**Exercises 6.1:** On Maximum Planarity:

1. Draw all regular maximum planar graphs, i.e., $\delta(G) = \Delta(G)$.

**Exercises 6.2:** On Outer Planarity: A graph is called outer planar if it can be embedded in the plane such that all vertices lie on the boundary of the unbounded area.

1. Let $G$ be outer planar, then prove that $e \leq 2n - 3$.
2. Show that $\chi(G) \leq 3$ if $G$ is outer planar.
3. Prove that a graph is outer planar if and only if it contains no subgraph homeomorphic with $K_4$ or $K_{3,2}$. 