PLANAR GRAPHS

5 Characterizing Planar Graphs

DEFINITION 5.1: Two graphs G and H are homeomorphic if one can be obtained from the other by insertion or deletion of vertices of degree two or a sequence of such operations.

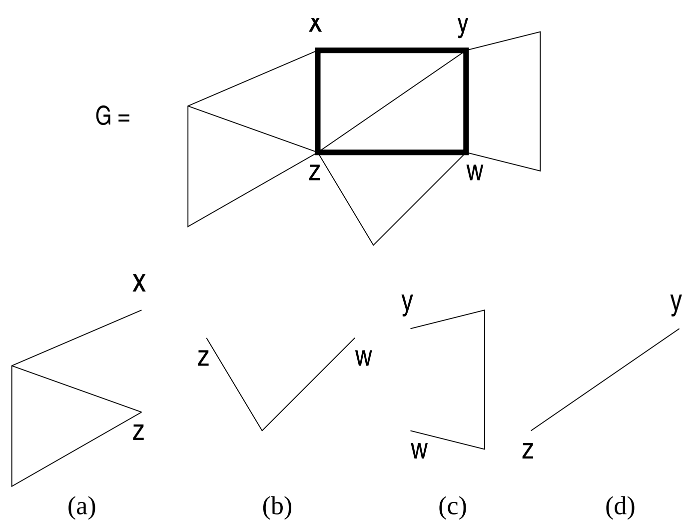

Fig. 3: Defining interior and exterior bridges

DEFINITION 5.2: Interior and exterior bridges: By example, consider Figure 3 that depicts a planar embedding of the graph G. A simple circuit $C = xywz$ is denoted by the prominent lines. The pieces $(a), (b), (c)$ and (d) are relative to C: If a piece has two or more points of contact with C it is a bridge. In this case, (d) is an interior bridge and $(a), (b)$ and (c) are exterior bridges.

Before we prove the celebrated Kuratowski theorem, one should notice the following. Since any planar graph can be embedded on a sphere, any area can be nominated the infinite area. Meaning that for any edge xy of a planar graph G, we can draw G in such a way that xy bounds the infinite area.

THEOREM 5.1 (Kuratowski (1930)): A graph is non planar if and only if it contains a subgraph homeomorphic to K_5 or $K_{3,3}$.

Proof: ($\Leftarrow$) K_5 and $K_{3,3}$ are non planar, so if G contains a subgraph homeomorphic to K_5or $K_{3,3}$ it too must be non planar.
($\Rightarrow$) Assume that the theorem is not true. Consider the set of counter examples with the smallest number of edges E. Let G be a member of this set with the least number of vertices. Denote $E(G) = N$, clearly G is non planar and does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$. For such a graph G the following statements hold:
(1) G is connected. Assume that G is disconnected, in which case it must consist of components $c_1, c_2, \ldots, c_k$ $(k \geq 2)$. Subsequently, $E(c_i)$ for each $i = 1, \ldots, k$ must be less than N (the case where $E(c_i) = 0$ for all c_i but one is covered by the fact that the order of G is minimal). Since G does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$, neither do c_i $(i = 1, 2, \ldots, k)$ and hence $c_1, c_2, \ldots, c_k$ are planar. If the components are planar then G too must be planar. Hence, by contradiction, G must be connected.

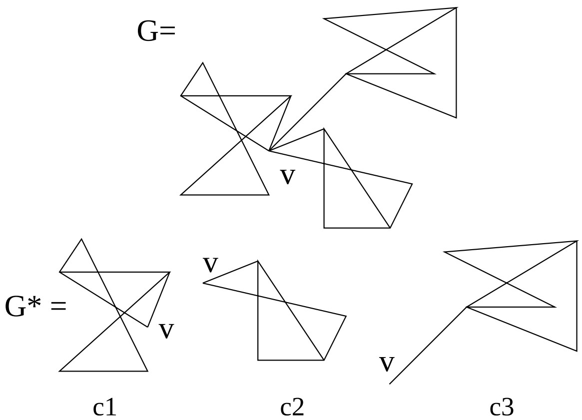

Fig. 4: G holds a cut vertex v

(2) G does not contain a cut vertex, meaning we cannot disconnect G by removing a single vertex v. Assume that G contains a cut vertex v and let G^* be G with a separation at v (see Figure 4). Since G^* is disconnected, it must consist of components c_i $(i = 1, 2, \ldots, n)$. Each c_i does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$, since G does not contain such a subgraph. Therefore, since $E(c_i) < N$ $(i = 1, 2, \ldots, n)$, each c_i is planar. Now, v can be moved, in each case, to be incident to the exterior area. In this way the components can be rejoined to give a planar G and so, by contradiction, G does not contain a cut vertex.

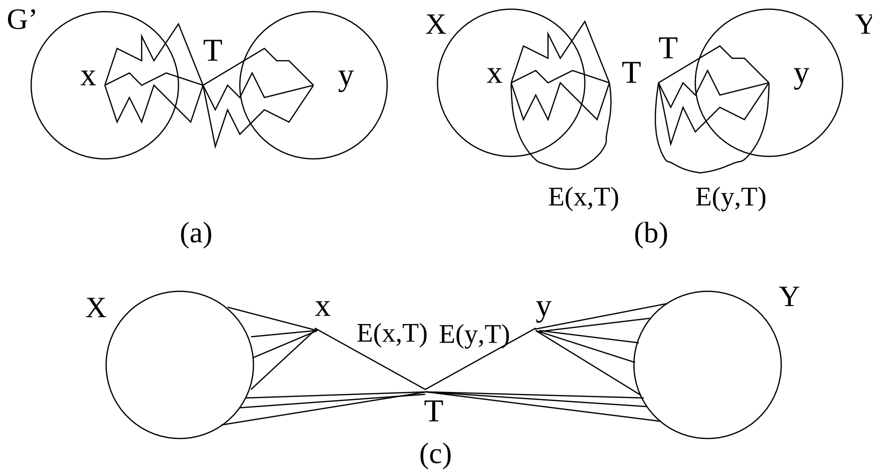

Fig. 5: G' holds a circuit through x and y

(3) Let $e(x, y)$ be the edge from x to y. Let G' be G with $e(x, y)$ omitted. G' contains a simple circuit through x and y. Indeed, since G contains no cut vertex (due to (2)), G' is connected (otherwise x would be a cut vertex). Now, assume that such a circuit does not exist, in which case a vertex, T say, exists of the form shown in Figure 5(a), see Menger's Theorem. Under separation X and Y are formed as shown in Figure 5(b) by adding an edge $e(x, T)$ and an edge $e(y, T)$. Since G contains a subgraph homeomorphic to X (because T is connected to y and $e(x, y)$ was part of G) and another homeomorphic to Y, X and Y do not contain subgraphs homeomorphic to K_5 or $K_{3,3}$. Moreover, $E(X) < N$ and $E(Y) < N$, so X and Y are planar. X and Y can be transformed such that $e(y, T)$ and $e(x, T)$ lie incident to the

exterior area and be brought together as shown in Figure 5(c). Now, deleting $e(x,T)$ and $e(y,T)$ and adding $e(x,y)$ must therefore give a plane representation of G. So, by contradiction, G must contain simple circuit from x to y.

We have now deduced: (i) $G' = G - e(x,y)$ is connected and contains a simple circuit C through x and y, (ii) G' contains no subgraph homeomorphic to K_5 or $K_{3,3}$, and (iii) Since $E(G) = N$ and $E(G') = N-1$, G' is planar.

Let G'_p be a planar embedding of G' and let C be a circuit through x and y such that it encloses as many areas as possible. Denote the path from x to y inclusive by $[x,y]$ and the path from x to y non inclusive by (x,y). No exterior bridge can have more than one point of contact with C in $[x,y]$ or $[y,x]$, or else a circuit C could be found that encloses at least one more area. Now, consider the interior and exterior bridges of G'_p with respect to C so that G will be non planar. There must be at least one exterior (E) and one interior (I) bridge or else G would be planar (by drawing $e(x,y)$ as an interior resp. exterior bridge). E must have contact points i and j with C, such that $i \in (x,y)$ and $j \in (y,x)$, and I must have at least 2 contact points a and b with C such that $a \in (x,y)$ and $b \in (y,x)$ (such a bridge I exists as $e(x,y)$ could otherwise be drawn as an interior bridge). The 4 possibilities which meet this criteria are given by Figure 6 (case (C) and (D) are equivalent).

The graph drawn in scenario A has a subgraph homeomorphic to $K_{3,3}$ (i.e., $\{i,b,x\},\{j,a,y\}$). In the other 4 scenarios the interior bridge I can still be drawn as an exterior bridge. We have chosen this interior bridge I such that it cannot be drawn as an exterior bridge without violating the planarity of G'_p (indeed, if all interior bridges could be drawn on the outside then drawing $e(x,y)$ as an interior bridge would make G planar), thus there is either another exterior bridge E' that prevents this, which implies that we end up in scenario A, with E' playing the role of E (because all exterior bridges have 1 contact point on (x,y) and 1 on (y,x)). Or I has at least one more contact point c that prevents us from drawing I as an outside bridge. Let us discuss these 4 scenarios one at a time.

In scenario C we need to add c on (j,i). However adding c on (x,i) would result in scenario A (where c plays the role of a); hence, we add c on $(j,x]$ (see Figure 7C). We know prove that this graph contains a subgraph homeomorphic to $K_{3,3}$. The bridge I contains a vertex v such that there are 3

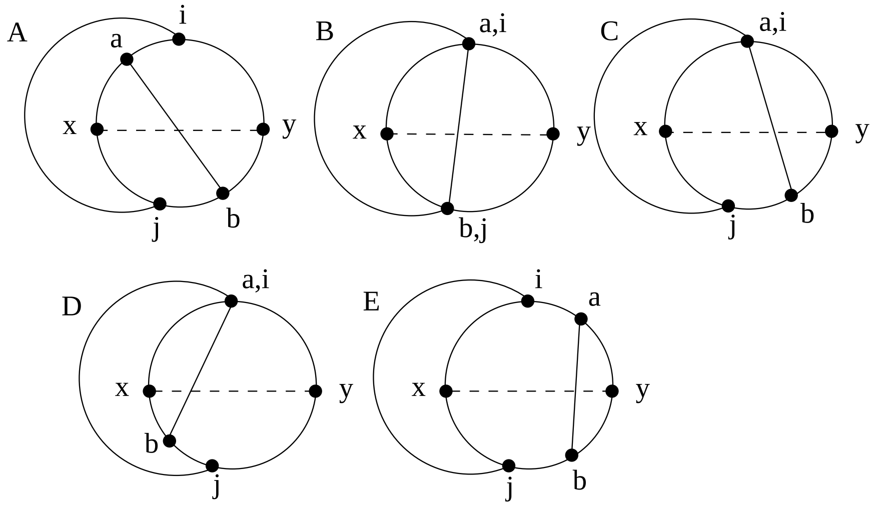

Fig. 6: 5 possible scenarios

vertex-distinct paths from v to a, b and c [4]. As a result, $\{a, b, c\}$ and $\{v, j, y\}$ form the vertices of a subgraph of G which is homeomorphic to $K_{3,3}$. Similarly, in scenario D we need to add c on $[y, j)$ and $\{a, b, c\}$ and $\{v, j, x\}$ form the vertices of a subgraph of G which is homeomorphic to $K_{3,3}$ (see Figure 7D). Also, in scenario E, c needs to coincide with x, otherwise if c was part of (j, x) or (x, i) and we end up in scenario A. As a result, $\{a, b, x\}$ and $\{v, y, i\}$ form the vertices of a subgraph of G which is homeomorphic to $K_{3,3}$ (see Figure 7E).

In scenario B, we need to add two contact points c and d (otherwise we could still draw I on the outside), one on (j, i) and one on (i, j). If either c or d do not coincide with x or y, we end up in scenario C or D. Thus, c has to coincide with x and d with y. We distinguish 2 cases: (i) there exists a vertex v in I such that there are 4 vertex-distinct paths from v to a, b, c and d (see Figure 7B2). In this case v, a, b, c and d form the vertices of a subgraph of G which is homeomorphic to K_5. (ii) if there is no such vertex v, then 2 vertices w_1 and w_2 exist[5] such that there are 5 vertex-distinct paths: one from w_1 to

[4] Indeed, let P be a path from a to b on I and w a vertex on this path, then setting v equal to the first common vertex of P and a path P' from c to w in $I - \{a, b\}$ suffices.

[5] Indeed, there exists a v such that there are vertex-distinct paths from v to a, b and c

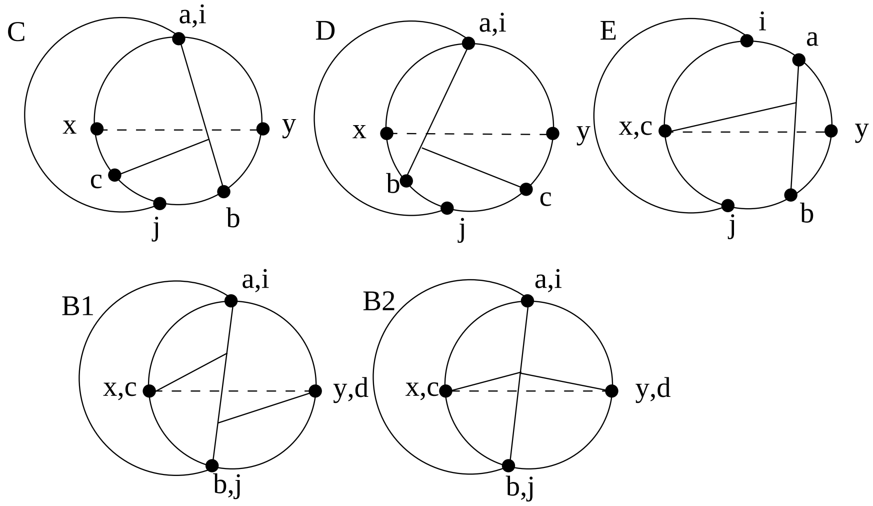

Fig. 7: 5 remaining cases

w_2, two from w_1 to a and c, and two from w_2 to b and d (see Figure 7B1). As a result, $\{w_1, b, d\}$ and $\{a, c, w_2\}$ form the vertices of a subgraph of G which is homeomorphic to $K_{3,3}$.

Q.E.D.

EXERCISES 5.1: On the Kuratowski Theorem:

1. Check whether the Petersen graph is planar.
2. Determine the number of non planar graphs G with 6 vertices.

6 Maximal Planar Graphs

DEFINITION 6.1: A maximal planar graph G is a planar graph to which no new edge can be added without violating the planarity of G. A triangulation is a planar graph G in which every area (region) is bounded by three edges.

THEOREM 6.1: The following statements are equivalent for a graph G with n_G vertices and e_G edges:

(in I). Now, there is also a path from d to c and G' is planar. Thus, thus the path from d to c has to connect to the path from a to b at some point. Without loss of generality we assume that this happens below v.

1. G is maximal planar.
2. G is a triangulation.
3. $e_G = 3n_G - 6$ and G is planar.

Proof: $(1 \Rightarrow 2)$ Suppose there is a planar representation of G that contains an area A bounded by 4 or more edges. Two vertices x and y incident to A exist such that $xy \notin E(G)$, otherwise we could add a vertex in the interior of A to obtain a planar representation of K_5. Drawing the edge xy in the interior of A does not violate the planarity of G.
$(2 \Rightarrow 3)$ G is a triangulation, therefore, $3f = 2e$. Applying Euler's equality for planar graphs we find $3n - e = 6$.
$(3 \Rightarrow 1)$ Earlier in the course, we saw that $e \leq 3n - 6$ for any planar graph G. Adding an edge thus violates the planarity of G.
Q.E.D.

Exercises 6.1: On Maximum Planarity:

1. Draw all regular maximum planar graphs, i.e., $\delta(G) = \Delta(G)$.

Exercises 6.2: On Outer Planarity: A graph is called *outer planar* if it can be embedded in the plane such that all vertices lie on the boundary of the unbounded area.

1. Let G be outer planar, then prove that $e \leq 2n - 3$.
2. Show that $\chi(G) \leq 3$ if G is outer planar.
3. Prove that a graph is outer planar if and only if it contains no subgraph homeomorphic with K_4 or $K_{3,2}$.